MATH 253 – WORKSHEET 17
LAGRANGE MULTIPLIERS

1. Optimization

1.1. **Ordinary optimization.** Suppose we want to find the maximum or minimum of $f(x,y)$ in a region R. We solve the system of equations $\vec{\nabla} f(x_0,y_0) = \vec{0}$ to find the *critical points*, and then evaluate f at critical points and on the boundary of R.

1.2. **Constrained optimization.** Suppose we want to find the maximum or minimum of $f(x,y)$ *subject to the constraint* $g(x,y)=0$. **Fact**: any local maximum/minimum *on the level set of* g occurs at a point (x_0,y_0) where $\vec{\nabla} f$ is proportional to $\vec{\nabla} g$. In other words, to find local maxima/minima we solve the system of equations

$$\begin{cases} \frac{\partial f}{\partial x}(x_0,y_0) & = \lambda \frac{\partial g}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) & = \lambda \frac{\partial g}{\partial y}(x_0,y_0) \\ g(x_0,y_0) & = 0 \end{cases}$$

where the unknowns are x_0, y_0, λ.

2. Problems

(1) Find the equation of the plane which passes through $(1,2,3)$ and encloses the smallest volume in the positive octant.

Solution: Suppose the plane meets the axes at $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ [parametrization step]. The volume of the enclosed pyramid is then $V(a,b,c) = \frac{1}{6}abc$ [express quantity in problem in terms of the parameters]. The equation of the plane through these points is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ [express object in the problem in terms of the parameters], and this passes through $(1,2,3)$ iff $\frac{1}{a}+\frac{2}{b}+\frac{3}{c}=1$ [express condition in the problem in terms of the paraemters]. We therefore need to minimize $V(a,b,c)$ subject go $g(a,b,c)=1$ where $g(a,b,c)=\frac{1}{a}+\frac{2}{b}+\frac{3}{c}$ [final restatement involving only mathematical functions]. By the method of Lagrange multipliers, at any local minimum there is λ such that

$$\begin{cases} \frac{\partial V}{\partial a} & = \lambda \frac{\partial g}{\partial a} \\ \frac{\partial V}{\partial b} & = \lambda \frac{\partial g}{\partial b} \\ \frac{\partial V}{\partial c} & = \lambda \frac{\partial g}{\partial c} \\ g(a,b,c) & = 1 \end{cases},$$

that is [convert problem to equations for the parameters]

$$\begin{cases} \frac{1}{6}bc & = -\frac{\lambda}{a^2} \\ \frac{1}{6}bc & = -\frac{2\lambda}{b^2} \\ \frac{1}{6}bc & = -\frac{3\lambda}{c^2} \\ \frac{1}{a}+\frac{2}{b}+\frac{3}{c} & = 1 \end{cases}.$$

[Now we solve the equation] First, $\lambda \neq 0$ since otherwise one of a,b,c would be zero which is impossible (the plane would pass through the origin and not enclose a finite volume). We now divide each equation by λ and multiply by $-a,-b,-c$ respectively to find

$$\frac{1}{a} = \frac{2}{b} = \frac{3}{b} = -\frac{abc}{6\lambda}.$$

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From the constraint we see that the three equal numbers $\frac{1}{a}, \frac{2}{b}, \frac{3}{c}$ add to 1, so each is equal to $\frac{1}{3}$. We conlcude that $a = 3$, $b = 6$, $c = 9$. [Finally, use the values of the parameters to solve the actual problem] It follows that the minimal volume is $\frac{1}{6}3 \cdot 6 \cdot 9 = 27$ and it occurs for the plane $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$.

(2) Find the absolute max and min of $f(x,y) = x^3y^2 - 2y^4x + 2x$ on $\left\{x^2 + y^2 \le 4\right\}$.

Solution: At the start we have an ordinary optimization problem, so we start by looking for critical points in the interior of the domain. These occur at (x,y) where:

$$\begin{cases} f_x = 3x^2y^2 - 2y^4 + 2 & = 0 \\ f_y = 2x^3y - 8y^3x & = 0 \end{cases}$$

Rewriting the second equation as $2xy\left(x^2 - 4y^2\right) = 0$ we see that at any critical point we either have $x = 0$ or $y = 0$ or $x = \pm 2y$. But $f_x\left(x, 0\right) = 2 \neq 0$ so no critical point has $y = 0$ and if $x^2 = 4y^2$ then $f_x(x,y) = 12y^4 - 2y^4 + 2 = 10y^4 + 2 \ge 2 > 0$ so no critical point has $x = \pm 2y$ either. But if $x = 0$ then $f_x = 0$ iff $2\left(1 - y^4\right) = 0$ iff $y = \pm 1$, so we have critical points at $(0, \pm 1)$. Since $f\left(0, y\right) = 0$, at both critical points the function vanishes.

Next, we consider the boundary. We need to minimize and maximize f on the circle $x^2 + y^2 = 4$. By the method of Lagrange multipliers, the maximum and minimum occur at points x, y where

$$\begin{cases} 3x^2y^2 - 2y^4 + 2 & = 2\lambda x \\ 2xy\left(x^2 - 4y^2\right) & = 2\lambda y \\ x^2 + y^2 & = 4 \end{cases}.$$

The second equation seems simplest so we start with it. If $y = 0$ we simply evaluate f to find $f(\pm 2, 0) = \pm 4$ (so we now see that the two critical points are neither the maximum nor the minimum). If $y \neq 0$ then the second equation implies

$$2x\left(x^2 - 4y^2\right) = 2\lambda\,.$$

Multiplying by x and combining with the first equation we get

$$3x^2y^2 - 2y^4 + 2 = 2x^2\left(x^2 - 4y^2\right)\,.$$

In order to solve this set $u = y^2$, so that $x^2 = 4 - u$. The equation then becomes

$$3\left(4 - u\right)u - 2u^2 + 2 = 2(4 - u)(4 - u - 4u)$$

or

$$15u^2 - 60u - 30 = 0\,.$$

It follows that $u^2 - 4u + 4 = 2$ or $u = 2 \pm \sqrt{2}$ and $y = \pm\sqrt{2 \pm \sqrt{2}}$. In each of those cases we have $x^2 = 4 - u = 2 \mp \sqrt{2}$ so we need to consider the eight points (x_0, y_0) of the form $\left(\pm\sqrt{2 - \sqrt{2}}, \pm\sqrt{2 + \sqrt{2}}\right)$ and $\left(\pm\sqrt{2 + \sqrt{2}}, \pm\sqrt{2 - \sqrt{2}}\right)$. Now $f(x,y) = x\left(x^2y^2 - 2y^4 + 2\right)$. In all cases we have $x_0^2y_0^2 = \left(2 + \sqrt{2}\right)\left(2 - \sqrt{2}\right) = 2$ and $y_0^2 = (2 \pm \sqrt{2})^2 = 4 + 2 \pm 4\sqrt{2} = 6 \pm 4\sqrt{2}$. It follows that

$$f\left(x_0, y_0\right) = x_0\left(4 - 6 \mp 4\sqrt{2}\right) = \pm 2\sqrt{2 \mp \sqrt{2}}\left(\pm 2\sqrt{2} + 1\right)$$

where the first sign is arbitrary and the next two are opposite. Numerical evaluation shows that the these points f takes the values $\approx \pm 5.8603$ and $\approx \pm 6.757$. It follows that the absolute maximum and minimum of f are $\pm 2\sqrt{2 + \sqrt{2}}\left(2\sqrt{2} - 1\right)$, attained at $\left(\pm\sqrt{2 + \sqrt{2}}, \pm\sqrt{2 - \sqrt{2}}\right)$ (the sign on x_0 determines if f is positive or negative, on y_0 doesn't matter).