MATH 253 – WORKSHEET 13
THE CHAIN RULE

(1) Define z as a function of x, y as the solution to $2x + 3y - 4z - e^{xyz-1} = 0$.

(a) Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: We differentiate the equation to get $2 - 4\frac{\partial z}{\partial x} - yze^{xyz-1} - xye^{xyz-1}\frac{\partial z}{\partial x} = 0$ and solve for z_x to get

$$\frac{\partial z}{\partial x} = \frac{2 - yze^{xyz-1}}{4 + xye^{xyz-1}}\,.$$

Similarly, $3 - 4\frac{\partial z}{\partial x} - xze^{xyz-1} - xye^{xyz-1}\frac{\partial z}{\partial y} = 0$ and hence

$$\frac{\partial z}{\partial y} = \frac{3 - xze^{xyz-1}}{4 + xye^{xyz-1}}\,.$$

Discussion: Let $F(x, y, z) = 2x+3y-4z-e^{xyz-1}$, and let $z = z(x, y)$ be the function implicitely defined by $F = 0$. Then the two-variable composite function $(x, y) \mapsto F(x, y, z(x, y))$ is the constant zero (that's how $z(x, y)$ is defined!). Its derivatives are therefore zero. But we can also calculate them using the chain rule:

$$\frac{\partial F(x, y, z(x, y))}{\partial x} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\,.$$

Since both methods for calculating the derivative must give the same answer (zero), we can solve for z_x:

$$z_x = -\frac{F_x(x, y, z(x, y))}{F_z(x, y, z(x, y))}\,.$$

(b) Find the plane tangent to this surface at $(1, 1, 1)$.

Solution: We verify that $(1, 1, 1)$ is on the surface: $2\cdot 1+3\cdot 1-4\cdot 1-e^{1\cdot 1\cdot 1-1} = 2+3-4-e^0 = 0$. Now at $(1, 1, 1)$ we have $\frac{\partial z}{\partial x}(1, 1) = \frac{2-1\cdot 1\cdot e^{1\cdot 1\cdot 1-1}}{4+1\cdot 1\cdot e^{1\cdot 1\cdot 1-1}} = \frac{1}{5}$ and $\frac{\partial z}{\partial y}(1, 1) = \frac{3-1\cdot 1\cdot e^{1\cdot 1\cdot 1-1}}{4+1\cdot 1\cdot e^{1\cdot 1\cdot 1-1}} = \frac{2}{5}$. It follows that the plane has the equation

$$z - 1 = \frac{1}{5}(x - 1) + \frac{2}{5}(y - 1)$$

or

$$5z - x - 2y = 2\,.$$

(c) Find an approximate solution to $\frac{5}{3} + \frac{7}{2} - 4z - e^{\frac{35}{36}z-1} = 0$.

Solution: We recognize the equation as $F\left(\frac{5}{6}, \frac{7}{6}, z\right) = 0$, that is as the equation defining $z\left(\frac{5}{6}, \frac{7}{6}\right)$. We can approximate this value by a linear approximation about $z(1, 1)$ (which we already know from part (b)). The linear approximation is

$$z(x, y) \approx 1 + \frac{1}{5}(x - 1) + \frac{2}{5}(y - 1)\,.$$

Plugging in $x = \frac{5}{6}$ and $y = \frac{7}{6}$ gives

$$z\left(\frac{5}{6}, \frac{7}{6}\right) \approx 1 + \frac{1}{5}\left(\frac{5}{6} - 1\right) + \frac{2}{5}\left(\frac{7}{6} - 1\right) = 1 - \frac{1}{30} + \frac{2}{30} = \frac{31}{30}\,,$$

(2) Suppose that $w = x^2 + yz - \ln(1 + z)$, that $x = st$, that $y = s + t$ and that $z = \frac{s}{t}$. Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

Solution: We have

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} = (2x)t + z\cdot 1 + \left(y - \frac{1}{1+z}\right)\frac{1}{t} = 2st^2 + \frac{s}{t} + \left(s + t - \frac{t}{s+t}\right)\frac{1}{t}\,.$$

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Similarly,

$$\frac{\partial w}{\partial s} = (2x)s + z \cdot 1 + \left(y - \frac{1}{1+z}\right)\left(-\frac{s}{t^2}\right) = 2s^2t + \frac{s}{t} - \left(s + t - \frac{t}{s+t}\right)\frac{s}{t^2}.$$

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(3) Suppose that z is a function of x, y and that x, y are functions of r, θ according to $x = r\cos\theta$, $y = r\sin\theta$. Express $\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$ in terms of $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution: We have $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$ and $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta)$. It follows that

$$\begin{aligned}
\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 &= (z_x\cos\theta + z_y\sin\theta)^2 + \frac{1}{r^2}\left[r\left(z_y\cos\theta - z_x\sin\theta\right)\right]^2 \\
&= z_x^2\cos^2\theta + z_y^2\sin^2\theta + 2z_xz_y\cos\theta\sin\theta + z_x^2\sin^2\theta + z_y^2\cos^2\theta - 2z_xz_y\cos\theta\sin\theta \\
&= z_x^2\left(\cos^2\theta + \sin^2\theta\right) + z_y^2\left(\cos^2\theta + \sin^2\theta\right) \\
&= z_x^2 + z_y^2.
\end{aligned}$$

(4) You are driving at a constant speed on a road that keeps a fixed compass direction as it goes over a hill. Say you position at time t is $(1-t, t)$, and hill is described by $z = e^{-x^2-y^2}$. How fast is your elevation changing at time t? When is your elevation maximal? What is it then?

Solution: Since $\frac{\partial x}{\partial t} = -1$, $\frac{\partial y}{\partial t} = 1$ we have $\frac{\partial z}{\partial t} = -2xe^{-x^2-y^2}(-1) - 2ye^{-x^2-y^2}(1) = 2(x-y)e^{-x^2-y^2}$, that is

$$\frac{\partial z}{\partial t} = 2(1-2t)e^{-\left(1+2t^2-2t\right)}.$$

This vanishes when $x - y = 0$ that is when $1 - 2t = 0$ or $t = \frac{1}{2}$, at which point the elevation is $e^{-\frac{1}{2^2}-\frac{1}{2^2}} = 1/\sqrt{e}$.