MATH 253 – WORKSHEET 2
VECTORS AND THE DOT PRODUCTS

1. Vectors and the Dot Product

(1) The wind is blowing eastward at 50km/h. A plane is travelling to the northeast. If the airspeed of the plane is 353km/h, what is its speed relative to the ground? [Note: $353 \cdot \frac{\sqrt{2}}{2} \approx 250$]?

Solution 1: Let $\vec{E}$, $\vec{N}$ be unit vectors in the directions east and north. Then the velocity of the plane relative to the wind is (all numbers in *kmph*) $250\vec{E} + 250\vec{N}$ (so that its total velocity is $\sqrt{2} \cdot 250 \approx 353$). The velocity of the wind is the vector $50\vec{E}$, so the velocity relative to the ground is $\left(50\vec{E}\right) + \left(250\vec{E} + 250\vec{N}\right) = 300\vec{E} + 250\vec{N}$, and the magnitude of this is $\sqrt{300^2 + 250^2} kmph$. With a calculator this turns out to be about $390kmph$.

Solution 2: In coordinates, the velocity of the plane is $\langle 250, 250\rangle$, the velocity of the wind is $\langle 50, 0\rangle$ and their sum is $\langle 300, 250\rangle$.

(2) What is the compass heading of the plane? (this is the angle between North and the direction of the plane).

Solution 1: This is $\arctan \frac{300}{250}$. With a calculator this is about 0.876 or about 50.2°.

Solution 2: Let θ be angle between the velocity $\vec{V} = 300\vec{E} + 250\vec{N}$ of the plane and north. Then

$$\cos\theta = \frac{\vec{V} \cdot \vec{N}}{\left|\vec{V}\right| \left|\vec{N}\right|} = \frac{300\vec{E} \cdot \vec{N} + 250\vec{N} \cdot \vec{N}}{\sqrt{300^2 + 250^2}} = \frac{250}{\sqrt{300^2 + 250^2}}$$

since $\vec{E} \perp \vec{N}$ while $\left|\vec{N}\right| = 1$ by definition. Thus $\theta = \arccos \frac{250}{\sqrt{300^2+250^2}}$.

(3) A pyramid is built from three $90° - 45° - 45°$ triangles, together with an equilateral triangle as base. What angle does a side make with the base?

Solution 1: Place the apex of the pyramid at the origin, so that its three sides are along the coordinate planes with the edges pointing along the coordinate axes. Then the vertices of the pyramid are $O = (0, 0, 0)$, $P = (1, 0, 0)$, $Q = (0, 1, 0)$ and $R = (0, 0, 1)$. The side OPQ and the base PQR intersect in the line $\overline{PQ}$. Let T be the midpoint of PQ. Then the vector $\overrightarrow{TO}$ is orthogonal to $\overline{PQ}$ at T, as is the vector $\overrightarrow{TR}$. Now the vector $\overrightarrow{OT}$ is the average of $\overrightarrow{OP} = \vec{i}$ and $\overrightarrow{OQ} = \vec{j}$ so $\overrightarrow{OT} = \frac{1}{2}\left(\vec{i} + \vec{j}\right)$ and $\overrightarrow{TO} = -\overrightarrow{OT}$. Also, $\overrightarrow{TR} = \overrightarrow{TO} + \overrightarrow{OR} = -\frac{1}{2}\vec{i} - \frac{1}{2}\vec{j} + \vec{k}$. Then the desired angle satisfies

$$\begin{aligned}
\overrightarrow{TO} \cdot \overrightarrow{TR} &= \left[-\frac{1}{2}\left(\vec{i} + \vec{j}\right)\right] \cdot \left[-\frac{1}{2}\left(\vec{i} + \vec{j}\right)\right] + \left[-\frac{1}{2}\left(\vec{i} + \vec{j}\right)\right] \cdot \vec{k} \\
&= \frac{1}{4}\left(\vec{i} \cdot \vec{i} + 2\vec{i} \cdot \vec{j} + \vec{j} \cdot \vec{j}\right) - \frac{1}{2}\left(\vec{i} \cdot \vec{k} + \vec{j} \cdot \vec{k}\right) \\
&= \frac{1}{4}(1 + 0 + 1) - \frac{1}{2}(0 + 0) \\
&= \frac{1}{2},
\end{aligned}$$

where we used the quadratic formula to calculate $\left(\vec{i} + \vec{j}\right) \cdot \left(\vec{i} + \vec{j}\right)$, and that $\vec{i}, \vec{j}, \vec{k}$ have magnitude 1 and are orthogonal to each other. It similarly follows that:

$$\cos\theta = \frac{\overrightarrow{TO} \cdot \overrightarrow{TR}}{\left|\overrightarrow{TO}\right| \left|\overrightarrow{TR}\right|} = \frac{\frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{1}{4} + \frac{1}{4}}\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} = \frac{2}{\sqrt{2}\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

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With a calculator one finds that $\theta \approx 0.955$ or about $54.7°$.

Solution 2: In coordinates we find that $T = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ so $\overrightarrow{TO} = \left\langle -\frac{1}{2}, -\frac{1}{2}, 0\right\rangle$ and $\overrightarrow{TR} = \left\langle -\frac{1}{2}, -\frac{1}{2}, 1\right\rangle$. Then $\left|\overrightarrow{TO}\right| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{\sqrt{2}}$, $\left|\overrightarrow{TR}\right| = \sqrt{\frac{1}{4} + \frac{1}{4} + 1} = \sqrt{3/2}$ and $\overrightarrow{TO} \cdot \overrightarrow{TR} = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + 0 \cdot 1 = \frac{1}{2}$. Thus

$$\cos\theta = \frac{1/2}{\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}}} = \frac{1}{\sqrt{3}}\,.$$