Math 223: Problem Set 11 (due 23/11/2012)

Practice problems

Section 6.1

PRAC Write down some matrix $A \in M_4(\mathbb{R})$ such that A has four distinct eigenvalues (your choice) with the correspoding eigenvectors being $\begin{pmatrix}1\\2\\0\\3\end{pmatrix}, \begin{pmatrix}2\\4\\1\\6\end{pmatrix}, \begin{pmatrix}2\\2\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\\2\end{pmatrix}$.

Real vs Complex eigenvalues

1. (a) Let V be a real vector space of odd dimension. Prove that every $T \in \mathrm{End}(V)$ has a real eigenvalue.
   (b) Define $T\colon \mathbb{R}[x]^{\leq 3} \to \mathbb{R}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Prove that T has no real eigenvalues. (Hint: what is T^2?)
   (c) Define $T\colon \mathbb{C}[x]^{\leq 3} \to \mathbb{C}[x]^{\leq 3}$ by $(Tp)(x) = x^3 p(-1/x)$. Find the spectrum of T and exhibit one eigenvector for each eigenvalue.

Commuting maps

2. Fix a vector space V and let $T, S \in \mathrm{End}(V)$ satisfy $TS = ST$.
   (a) Suppose that $T\underline{v} = \lambda\underline{v}$ for some λ and $\underline{v} \in V$. Show that $T(S\underline{v}) = \lambda(S\underline{v})$.
   CONCLUSION Let $V_\lambda = \{\underline{v} \in V \mid T\underline{v} = \lambda\underline{v}\}$. Then $S(V_\lambda) \subset V_\lambda$.
   (b) Let $H = -\frac{d^2}{dx^2} + M_{x^2}$ be the operator on functions on $\mathbb{R}$. associated to the quantum harmonic oscillator, and let P be the operator of reflection at the origin ($(Pf)(x) = f(-x)$). Explain why we can assume that a basis of eigenfunctions of H consists of functions of *definite parity* (i.e. either even or odd).

Inner products and norms

3. The *trace* of a square matrix is the sum of its diagonal entries ($\mathrm{tr}A = \sum_{i=1}^n a_{ii}$).
   PRAC Show that $\mathrm{tr}\colon M_n(\mathbb{R}) \to \mathbb{R}$ is a linear functional.
   (a) Show that for any two square matrices A, B we have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.
   (**b) Find three $2x2$ matrices A, B, C such that $\mathrm{tr}(ABC) \neq \mathrm{tr}(BAC)$.
   (c) Show that $\mathrm{tr}(S^{-1}AS) = \mathrm{tr}(A)$ if S is invertible.
   PRAC Show that $\langle A, B\rangle \stackrel{\text{def}}{=} \mathrm{tr}(A^t B)$ is an inner product on $M_n(\mathbb{R})$
   DEF For $A \in M_{m,n}(\mathbb{C})$, its *Hermitian conjuate* is the matrix $A^\dagger \in M_{n,m}(\mathbb{C})$ with entries $a^\dagger_{ij} = \overline{a_{ji}}$ (complex conjuguate).
   (d) Show that $\langle A, B\rangle \stackrel{\text{def}}{=} \mathrm{tr}\left(A^\dagger B\right)$ is a Hermitian product on $M_n(\mathbb{C})$.

DEFINITION. Let V be a real or complex vector space. A *norm* (="notion of length") on V is a map $\|\cdot\|\colon V \to \mathbb{R}_{\geq 0}$such that

(1) $\|a\underline{v}\| = |a|\,\|\underline{v}\|$ (that is, $3\underline{v}$ is three times as long as $\underline{v}$)
(2) $\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| + \|\underline{v}\|$ ("triangle inequality")
(3) $\|\underline{v}\| = 0$ iff $\underline{v} = \underline{0}$ (note that one direction follows from (1)).

4. (Examples of norms)

(a) Show that $\|\underline{x}\|_\infty = \max_i |x_i|$ is a norm on $\mathbb{R}^n$ or $\mathbb{C}^n$.

(b) Show that $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$ is a norm on $C(a,b)$ (continuous functions on the interval $[a,b]$).

(c) (Sobolev norm) Show that $\|f\|_{H^1}^2 = \int_a^b \left(|f(x)|^2 + |f'(x)|^2 \right) \mathrm{d}x$ defines a norm on $C^\infty(a,b)$ (Hint: this norm is associated to an inner product)

Supplementary problem: the minimal polynomial

A. (Division with remainder) Let $p, a \in \mathbb{R}[x]$ with a non-zero. Show that there are unique $q, r \in \mathbb{R}[x]$ with $\deg r < \deg a$ such that $p = qa + r$. (Hint: let r be an element of minimal degree in the set $\{p - aq \mid q \in \mathbb{R}[x]\}$).

B. Let $A \in M_n(\mathbb{R})$.

(a) Show that there exists a non-zero $p \in \mathbb{R}[x]^{\leq n^2}$ such that $p(A) = 0$.

DEF A polynomial is *monic* if the highest-degree monomial has coefficient 1 ($x^2 + 3$ is monic, $2x^2 + 3$ is not).

(b) Rescaling the polynomial, show that there exists a monic polynomial p' of the same degree as p such that $p'(A) = 0$.

(c) Let $m_A \in \mathbb{R}[x]$ be a monic polynomial of minimal degree such that $m_A(A) = 0$. Let p be any polynomial such that $p(A) = 0$. Show that m divides p.

(d) Let m'_A be another monic polynomial of the same degree as m_A such that $m'_A(A) = 0$. Show that $m'_A = m_A$ (Hint: what is the degre of the difference?)

DEF m_A is called the *minimal polynomial* of A (saying “the” minimal polynomial is justified by part c).

(e) Conversely, show that if $p = m_A q$ for some $q \in \mathbb{R}[x]$ then $p(A) = 0$. Conclude that $\{p \in \mathbb{R}[x] \mid p(A) = 0\} = m_A \mathbb{R}[x] = \{m_A q \mid q \in \mathbb{R}[x]\}$.

RMK The Cayley–Hamilton Theorem states that $p_A(A) = 0$. It follows that $\deg m_A \leq n$ and that $m_A | p_A$.

Supplementary problem: The Rayleigh quotient

C. Given a matrix $A \in M_n(\mathbb{R})$ consider the function $f \colon \mathbb{R}^n \to \mathbb{R}$ given by $f(\underline{x}) = \underline{x}^t A \underline{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$. We introduce the notation $\|\underline{x}\|_2^2 = \sum_{i=1}^{n} x_i^2$.

(a) Show that $(\nabla f)(\underline{x}) = A\underline{x} + A^t \underline{x}$.

(b) Let $\underline{v}$ be the point where f attains its maximum on the unit sphere $S^{n-1} = \{\underline{x} \in \mathbb{R}^n \mid \|\underline{x}\| = 1\}$. Use the method of Largrange multipliers to show that $\underline{v}$ satisfies $A\underline{v} + A^t \underline{v} = \lambda \underline{v}$ for some $\lambda \in \mathbb{R}$.

(c) A matrix is *symmetric* if $A = A^t$. Show that every symmetric matrix has a real eigenvalue.

(d) Show that the following two maximization problems are equivalent:

$$\max\{f(\underline{x}) \mid \|\underline{w}\|_2 = 1\} \leftrightarrow \max\left\{\frac{f(\underline{x})}{\|\underline{x}\|_2^2} \mid \underline{x} \neq \underline{0}\right\}.$$