Math 223: Problem Set 3 (due 26/9/12)

Practice problems (recommended, but do not submit)

Section 1.6, Problems 1 (except (g)), 2-5, 7, 11,12, 22*, 24*.

Bases and dimension

1. (§1.6 E8)Let $W = \left\{\underline{x} \in \mathbb{R}^5 \mid \sum_{i=1}^{5} x_i = 0\right\}$ be the set of vectors in $\mathbb{R}^5$ whose co-ordinates sum to zero. The following 8 vectors span W. Find a subset of them which forms a basis for W.
   $\underline{u}_1 = (2,-3,4,-5,2)$, $\underline{u}_2 = (-6,9,-12,15,-6)$, $\underline{u}_3 = (3,-2,7,-9,1)$, $\underline{u}_4 = (2,-8,2,-2,6)$, $\underline{u}_5 = (-1,1,2,1,-3)$, $\underline{u}_6 = (0,-3,-18,9,12)$, $\underline{u}_7 = (1,0,-2,3,-2)$, $\underline{u}_8 = (2,-1,1,-9,7)$.

2. Find a basis for the subspace $\left\{\underline{x} \in \mathbb{R}^4 \mid x_1 + 3x_2 - x_3 = 0\right\}$ of $\mathbb{R}^4$. What is the dimension?

3. For $n \geq 0$ let $E = \{p \in \mathbb{R}[x] \mid \deg(p) \leq n,\ p(-x) = p(x)\}$ be the set of even polynomials of degree at most n. This is a subspace of $\mathbb{R}[x]$ (we will show that later). Find a basis for it.

Linear Functionals

Fix a vector space V. A *linear functional* on V is a map $\varphi\colon V \to \mathbb{R}$ such that for all $a,b \in \mathbb{R}$ and $\underline{u},\underline{v} \in V$, $\varphi(a\underline{v} + b\underline{u}) = a\varphi(\underline{v}) + b\varphi(\underline{u})$. Let $V^* \stackrel{\text{def}}{=} \left\{\varphi \in \mathbb{R}^V \mid \varphi \text{ is a linear functional}\right\}$ be the set of linear functionals on V. The space V^* is called the *dual vector space* of V.

4. (The basic example)
   (a) Show that $\varphi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x - 2y + 3z$ defines a linear functional on $\mathbb{R}^3$.
   (b) Let φ be a linear functional on $\mathbb{R}^2$. Show that $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x \cdot \varphi\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y \cdot \varphi\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ and conclude that every linear functional on $\mathbb{R}^2$ is of the form $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = ax + by$ for some $a,b \in \mathbb{R}$.
   SUPP Construct an identification of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$.

SUPP Show that V^* is a subspace of $\mathbb{R}^V$, hence a vector space.

6. Let V be a vector space and let $\varphi \in V^*$ be non-zero.
   (a) Show that $\operatorname{Ker}\varphi \stackrel{\text{def}}{=} \{\underline{v} \in V \mid \varphi(\underline{v}) = 0\}$ is a subspace.
   (*b) Show that there is $\underline{v} \in V$ satisfying $\varphi(\underline{v}) = 1$.
   (**c) Let B be a basis of $\operatorname{Ker}\varphi$, and let $\underline{v} \in V$ be as in part (b). Show that $B \cup \{\underline{v}\}$ is a basis of V.
   RMK If V is finite-dimensional this shows: $\dim V = \dim \operatorname{Ker}\varphi + 1$. In general we say that $\operatorname{Ker}\varphi$ is of *codimension* 1.

A Linear Transformation

In this problem our choice of letters to denote numbers follows conventions from physics. Thus v will be a numerical parameter rather than a vector, and we write the coordinates of a vector in $\mathbb{R}^2$ as $\begin{pmatrix} x \\ t \end{pmatrix}$ rather than $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

7. In the course of his researches on electromagnetism, Hendrik Lorentz wrote down the following map $L_v \colon \mathbb{R}^2 \to \mathbb{R}^2$:

$$L_v \begin{pmatrix} x \\ t \end{pmatrix} \stackrel{\text{def}}{=} \gamma_v \cdot \begin{pmatrix} x - vt \\ t - vx \end{pmatrix}.$$

Here v is a parameter such that $|v| < 1$ and γ_v is also a scalar, defined by $\gamma_v = \left(1 - v^2\right)^{-1/2}$.

(a) Suppose $v = 0.6$ so that $\gamma_v = (1 - 0.6^2)^{-1/2} = 1.25$. Calculate $L_v \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $L_v \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $L_v \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Check that $L_v \begin{pmatrix} 2 \\ 3 \end{pmatrix} = L_v \begin{pmatrix} 3 \\ 2 \end{pmatrix} + L_v \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) Show that L_v is a linear transformation.

(c) ("Group property") Let $v, v' \in (-1, 1)$ be two parameters. Show that $L_v \circ L_{v'} = L_u$ for $u = \frac{v+v'}{1+vv'}$. It is a fact that if $v, v' \in (-1, 1)$ then $\frac{v+v'}{1+vv'} \in (-1, 1)$ as well.
Hint: Start by showing $\gamma_v \gamma_{v'} = \frac{\gamma_u}{1+vv'}$.

RMK If $g \colon A \to B$ and $f \colon B \to C$ are functions then $f \circ g$ denotes their *composition*, the function $f \circ g \colon A \to C$ such that $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

Supplementary problems

A. Let V be a vector space and let $W_1, W_2 \subset V$ be finite-dimensional subspaces.

(a) Show that $\dim(W_1 + W_2) \le \dim W_1 + \dim W_2$.

(**b) Show that $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$.

RMK Let A, B be finite sets. Then the "inclusion-exclusion" formula states $\#A + \#B = \#(A \cup B) + \#(A \cap B)$

B. Let V be a vector space, W a subspace. Let $B \subset W$ be a basis for W and let $C \subset V$ be disjoint from B and such that $B \cup C$ is a basis for V (that is, we extend B until we get a basis for V).

(a) Show that $\{\underline{v} + W\}_{\underline{v} \in C}$ is a basis for the quotient vector space V/W (V/W is defined in the supplement to PS2).

(b) Show that $\dim W + \dim(V/W) = \dim V$.