## 1. Linear dependence

A – Definition of linear independence.

- $\underline{v}$ depends on $S$ if there are $\{\underline{s}_i\}_{i=1}^n \subset S$ and $\{a_i\}_{i=1}^n \subset \mathbb{R}$ such that $\underline{v} = \sum_{i=1}^n a_i \underline{s}_i$.
- $\underline{v}$ depends on $S$ if there are $\{\underline{s}_i\}_{i=1}^n \subset S$ and $\{a_i\}_{i=1}^n \subset \mathbb{R}$ not all zero such that $\underline{v} = \sum_{i=1}^n a_i \underline{s}_i$.
- If $\underline{v}$ depends on $S$ then there are $\{\underline{s}_i\}_{i=1}^n \subset S$ and $\{a_i\}_{i=1}^n \subset \mathbb{R}$ such that $\underline{v} = \sum_{i=1}^n a_i \underline{s}_i$.

B – Linear dependence of the zero vector. The problem is to decide if $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ depends on $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^2$.

$$\begin{aligned} a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ a+2b &= 0 \\ a &= -2b \\ b &= 0 \\ a &= 0 \end{aligned}$$

dependent

| | |
|---|---|
| Suppose that there were $a, b$ such that $a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ | Dependence would follow from the existence of $a, b$ such that $a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ |
| Then $a + 2b = 0$ | This is equivalent to $\begin{pmatrix} a+2b \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, hence to $a + 2b = 0$ |
| Thus $a = -2b$. | And hence equivalent to the existence of $a, b$ such that $a = -2b$. |
| Thus if there were $a, b$ and $b = 0$ then $a = 0$ also. | Choosing $b = 0, a = 0$ this equality holds so such $a, b$ do exist. |

## 2. Linear maps on $\mathbb{R}^n$

A – Linearity in a concrete example.

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map, and suppose that $T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \underline{u}$ and $T\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \underline{v}$. Find an explicit vector $\underline{x} \in \mathbb{R}^2$ such that $T\underline{x} = 2\underline{u} - 3\underline{v}$.

- $T\underline{x} = 2\underline{u} - 3\underline{v} = 2T\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3T\begin{pmatrix} 5 \\ 4 \end{pmatrix} = T\left(2\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 5 \\ 4 \end{pmatrix}\right) = T\begin{pmatrix} -9 \\ 10 \end{pmatrix}$. Therefore $T\underline{x} = T\begin{pmatrix} -9 \\ 10 \end{pmatrix}$. Therefore $\underline{x} = \begin{pmatrix} -9 \\ 10 \end{pmatrix}$.
- $2\underline{u} - 3\underline{v} = 2T\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3T\begin{pmatrix} 5 \\ 4 \end{pmatrix} = T\left(2\begin{pmatrix} 3 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 5 \\ 4 \end{pmatrix}\right) = T\begin{pmatrix} -9 \\ -10 \end{pmatrix}$ so $T\begin{pmatrix} -9 \\ -10 \end{pmatrix} = 2\underline{u} - 3\underline{v}$.

B – Linearity of a linear functional. OK
C – Definition of Kernel. OK
D – Basis.

- Suppse $\underline{v} \in \operatorname{Ker}\varphi$ and that $\underline{v} = a\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + b\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 5a+3b \\ 2a \\ -2b \end{pmatrix}$. Then $\varphi\underline{v} = 2(5a + 3b) - 5(2a) + 3(-2b) = 0$. Thus any vector $\underline{v} \in \operatorname{Ker}\varphi$ can be written as a linear combination of $\begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$ so they span $\operatorname{Ker}\varphi$.

## 3. Linear maps on $\mathbb{R}^{\mathbb{R}}$

For $(T_a f)(x) = f(x+a)$, $W = \text{Span}\{e^{rx} \mid r \in \mathbb{R}\}$.
B – Image of a linear map.
Show that $T_a W = W$.

- $T_a(e_r) = e_r(a)e_r \in W$ so $T_a W = W$.