TIPS FOR MATH 100

1. General

Reading comprehension. If the problem asks "express the following limit as the derivative of a function", you don't need to *evaluate* the derivative, just to name the funnction. [Quiz 3] If the problem asks "use the definition of the derivative", a solution using differentiation laws isn't enough (but is a good way to check your work!).

2. Limits

How to use limit laws. It is natural to expect that limits behave in certain ways. They actually do (we read about this in the beginning of the term). If asked to **justify** limit calculations here's an example:

$$\begin{aligned}
\lim_{x\to 3} \frac{\cos(3e^x)+x^5}{2x-1} &\overset{\text{ratio rule}}{=} \frac{\lim_{x\to 3}\left(\cos(3e^x)+x^5\right)}{\lim_{x\to 3}(2x-1)} \\
&\overset{\text{sum rum}}{=} \frac{\lim_{x\to 3}\cos(3e^x)+\lim_{x\to 3}x^5}{\lim_{x\to 3}(2x)-1} \\
&\overset{\text{continuity of cos}}{=} \frac{\cos\left(\lim_{x\to 3}(3e^x)\right)+\left(\lim_{x\to 3}x\right)^5}{\lim_{x\to 3}(2x)-1} \\
&\overset{\text{continuity of } e^x}{=} \frac{\cos\left(3e^{\lim_{x\to 3}x}\right)+\left(\lim_{x\to 3}x\right)^5}{\lim_{x\to 3}(2x)-1} \\
&= \frac{\cos(3e^3)+3^5}{5}.
\end{aligned}$$

3. Continuity

How to justify continuity 1. If a function is defined piecewise, recall the definition: f is continuous at a if $\lim_{x\to a} f(x) = f(a)$, so to justify continuity you need to

(1) Evaluate the limit
(2) Evaluate the function at a
(3) Check if the two are equals

It is not enough to check that left and rigit limits agree – unless you also check the value of the function.

How to justify continuity 2. If a function is given by a formula, use the limit laws: if f, g are continuous so are $f \pm g$, fg, $\frac{f}{g}$, $f(g(x))$ provided they make sense (e.g. provided we are not dividing by zero).

How to use the Intermediate Value Theorem.

(1) Set up a single function which needs to attain a single value. Commonly you this means subtracting the two sides of an equation and trying to make the difference zero:

   (a) To solve the equation $x^3 + \sin x = 100 + \sqrt{e^x + 1}$ consider $f(x) = x^3 + \sin x - \sqrt{e^x + 1} - 100$ and look for x such that $f(x) = 0$.
   (b) To solve the equation $f(x) = x$ consider the function $g(x) = f(x) - x$ and
   (c) To solve the "monk" problem, suppose that $f(t), g(t)$ describe the altitude of the monk at time t on days $1, 2$. Then we want t for which $f(t) = g(t)$, but to do this we consider the function $h(t) = f(t) - g(t)$. Then $h(7\text{am}) = 0 - H = -H$ and $h(7\text{pm}) = H - 0 = +H$ where H is the height of the mountain.
(2) Show that the function is continuous (see paragraph above)
(3) Check the values at two points.
   (a) If an interval is given, try the endpoints of the interval [e.g. monk problem]
   (b) If not, look for the behaviour for x large and small – try to see if one piece of the function "wins" and controls the sign [e.g. Quiz 2]
   (c) Make sure the function values at the endpoints do bracked the desired value.
(4) Mention IVT.
   (a) Don't forget to mention continuity and evaluate at endpoints first.
(5) Endgame: go back from vanishing of the new function to the original equation.

4. Computing limits and recognizing derivatives

Two examples on how to use $\lim_{u \to 0} \frac{\sin u}{u} = 1$. First we interpret this: for small u, can replace $\sin u$ with u. How does that play out in practice?

Example 1. Calculate $\lim_{x \to 0} \frac{\tan(ax)}{\sin(bx)}$. Well, $\tan(ax) = \frac{\sin(ax)}{\cos(ax)}$ and $\cos(ax)$ is well-behaved at 0. How do we replace $\sin(ax)$ with ax? By doing this and then paying back the price – this looks like the following:

$$\begin{aligned} \frac{\tan(ax)}{\sin(bx)} &= \frac{1}{\cos(bx)} \frac{\sin(ax)}{\sin(bx)} \\ &= \frac{1}{\cos(bx)} \frac{ax}{bx} \cdot \frac{\sin(ax)}{(ax)} \Big/ \frac{\sin(bx)}{(bx)} \end{aligned}$$

We note apply the quotient and product rules (recall the section above) to get

$$\begin{aligned} \lim_{x \to 0} \frac{\tan(ax)}{\sin(bx)} &= \frac{1}{\lim_{x \to 0} \cos(bx)} \cdot \frac{a}{b} \cdot \left[\lim_{x \to 0} \frac{\sin(ax)}{(ax)} \right] \Big/ \left[\lim_{x \to 0} \frac{\sin(bx)}{(bx)} \right] \\ &= \frac{1}{\cos(b \cdot 0)} \cdot \frac{a}{b} \cdot 1/1 \\ &= \frac{a}{b} \end{aligned}$$

Since if $x \to 0$ then $ax \to 0$.

Example 2. Compute $\lim_{\theta \to 0} \frac{\sin^2 \theta}{(\theta^2 + 7\theta)(e^\theta - 1)}$. Again we replace $\sin \theta$ with θ to get:

$$\begin{aligned} \frac{\sin^2 \theta}{(\theta^2 + 7\theta)(e^\theta - 1)} &= \frac{\sin^2 \theta}{\theta^2} \cdot \frac{\theta^2}{(\theta^2 + 7\theta)(e^\theta - 1)} \\ &= \left(\frac{\sin \theta}{\theta} \right)^2 \cdot \frac{1}{\theta + 7} \cdot \frac{\theta}{e^\theta - 1}. \end{aligned}$$

By the product rule

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{(\theta^2+7\theta)(e^\theta-1)}=\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)^2\frac{1}{\lim_{\theta\to 0}(\theta+7)}\cdot\lim_{\theta\to 0}\frac{\theta}{e^\theta-1}\,.$$

We recognize the first limit is 1, the second is $\frac{1}{7}$. For the last we recognize that $\lim_{\theta\to 0}\frac{e^\theta-e^0}{\theta}=f'(0)$ where $f(\theta)=e^\theta$. But then $f'(\theta)=e^\theta$as well so $f'(0)=e^0=1$ (this is the definition of e) and we find

$$\lim_{\theta\to 0}\frac{\sin^2\theta}{(\theta^2+7\theta)(e^\theta-1)}=(1)^2\frac{1}{7}\cdot\frac{1}{1}=\frac{1}{7}\,.$$