Math 121: Single-variable Calculus II
Lecture Notes

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These are rough notes for the spring 2012 course. Problem sets and solutions were posted on an internal website.

Contents

Introduction (Lecture 1, 4/1/12)

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0.1. Administrivia

Syllabus posted online, summarized on slides. Key points:

- Problem sets will be posted on the course website. Solutions will be posted on a secure system (email explanation will be sent).
  - Depending on time, the grader may only mark selected problems. Solutions will be complete.
- Absolutely essential to
  - Read ahead according to the posted schedule. Lectures after the first will assume that you had done your reading.
  - Do homework.
- Office hours
- Course website is important. Contains notes, problem sets, announcements, reading assignments etc.

0.2. Course plan (subject to revision)

Two tracks: foundations and technique. Topics:

- The Riemann integral
  - The problem of area; examples.
  - Construction of the integral, basic properties.
  - Techniques of integration
  - Applications
- Parametric curves and polar co-ordinates.
- The real numbers
  - Sequences and convergence.
  - Series.
  - Power series and Taylor expansion.

0.3. Motivating problem: the area of the disc

- What is the area of the disc of radius R? It is πR^2.
- How do we know?
  - Cut it up in pieces and add up the areas.

- Approximate slices by triangles
- Total bases approximates circumference
- Can also get upper bound.
- Archimedes: Method of “divide and sum”.
- Newton: Calculus.
- Weierstraß: rigorous bounds.

CHAPTER 1

The Riemann integral

1.1. Two preliminary tools: The Σ notation and induction (Lectures 2,3, 6-7/1/12)

- Plan:
  (1) Sequences and sums
    (a) Sequences and parametrization
    (b) Sums
  (2) Proof by Induction
- Goals: Sequences
  - Be able to parametrize sequences given the first few elements
  - Be able to do change-of-variable
  - Convert $\cdots$ notation to parametrization and vice versa
- Goals: sums
  - Be able to convert between $\cdots$ notation and Σ
- Goals: induction
  - Prove statements about sums using induction.
- Hook: You have won the lottery; choose between \$12.5M today and \$1M/year for 25 years.

1.1.1. Sequences.

DEFINITION 1. A *sequence* (properly, an infinite sequence) is a function whose domain is the natural numbers.

NOTATION 2. The value at $i \in \mathbb{N}$ (called the "ith element" of the sequence) is by the subscript i.

Math 121: In-class worksheet for lecture 2

EXAMPLE 3. Some explicit sequences:

(1) Constant Sequences $0,0,0,0,0,\cdots$ and $1,1,1,1,1,\cdots$
(2) The *natural numbers* $\mathbb{N}$: $0,1,2,3,4,\cdots$
(3) The *positive integers* $\mathbb{Z}_{\geq 1}$ $1,2,3,4,\cdots$
(4) An *arithmetic progression* $5,9,13,17,21,25,\cdots$
(5) A *geometric progression* $1,2,4,8,16,32,64\cdots$
(6) The *prime numbers* $2,3,5,7,11,13,\cdots$

EXERCISE 4. Parametrize the sequences above.

(1) $a_i = 0$ for $i \geq 0$ and $a_i = 1$ for $i \geq 0$.
(2) $b_j = j$ for $j \geq 0$.

(3)

(4)

(5)

(6)

EXERCISE 5. Parametrize the sequences

(1) $\sqrt{13}, \sqrt{20-\frac{1}{\pi}}, \sqrt{27-\frac{1}{\pi^2}}, \sqrt{34-\frac{1}{\pi^3}}, \cdots$
(2) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \cdots$

EXAMPLE 6. Simple sums

(1) $5+6+7+8+9=\sum_{i=5}^{9} i=\sum_{j=0}^{4}(5+j)=\sum_{j=1}^{5}(j+4)$
(2) $5+6+7+8+\cdots+150=\sum_{i=5}^{150} i$
(3) $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}=$
(4) $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\cdots$ (n terms)

- Key idea: *parametrizing* a sequence;

1.1.2. Sums.

EXAMPLE 7. $1 = 1$, $1+2 = 3$, $1+2+3 = 6$, $1+2+3+4 = 10$, $1+2+3+4+5 = 15$. $1+2+3+\cdots+n = ?$

NOTATION 8. Let $\{a_i\}_{i=0}^{\infty}$ be a sequence. Let m, M be integers. We write

$$\sum_{i=m}^{M} a_i$$

for the sum $a_m + a_{m+1} + \cdots + a_M$ of the elements on the sequence in positions between m, M.

- If $m = M$ there is only one summand, by convention the sum is equal to a_m.
- If $m > M$ the sum is empty, by convention it is zero.

LEMMA 9. $\sum_{i=k}^{l} a_i + \sum_{i=l+1}^{m} a_i = \sum_{i=k}^{m} a_i$; $c \sum_{i=m}^{M} a_i = \sum_{i=m}^{M} c a_i$.

1.1.3. Induction.

LEMMA 10. *For all natural numbers $n \geq 0$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q-1}$.*

PROOF. Both are true for $n = 0$. Consider $\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$. For the second, Problem set 1. $\square$

- Practical way the proof works.
- Statement about sets of integers.

EXAMPLE 11. Present value of discrete income stream (see example problem set I)

PROPOSITION 12. *Let $x \in \mathbb{R}$ satisfy $x \geq -1$. Then for all $n \in \mathbb{N}$we have $(1+x)^n \geq 1 + nx$.*

PROOF. For $n = 0$ both sides are equal to 1. Now assume that

$$(1+x)^n \geq 1 + nx .$$

We may then multiply both sides by $1 + x \geq 0$ to get:

$$\begin{aligned} (1+x)^{n+1} &\geq (1+nx)(1+x) \\ &= 1 + nx + x + nx^2 \\ &= 1 + (n+1)x + nx^2 \\ &\geq 1 + (n+1)x \end{aligned}$$

since $n \geq 0$ and $x^2 \geq 0$. $\square$

1.2. Examples

1.2.1. The area of a right triangle. (Lecture 4, 10/1/2011).

PROBLEM 13. Let $a, b > 0$. Let R be the triangle with vertices $(0,0)$, $(a,0)$, (a,b). What is its area?

Four steps:

(1) **Subdivision**: Partition $[0,a]$ into n subintervals, $\left[\frac{i-1}{n}a, \frac{i}{n}a\right]$ of length $\Delta x = \frac{a}{n}$ each. To each associate the strip $\left\{\frac{i-1}{n} \leq x \leq \frac{i}{n}a\right\}$.

(2) **Approximation**: The triangle intersects the ith strip in a trapezoidal region, containing the rectangle $\left[\frac{i-1}{n}a, \frac{i}{n}a\right] \times \left[0, \frac{i-1}{n}b\right]$ and contained in $\left[\frac{i-1}{n}a, \frac{i}{n}a\right] \times \left[0, \frac{i}{n}b\right]$.

(3) **Sum**: Summing the contribution from each strip, the area of the triangle lies between

$$\sum_{i=1}^{n} \Delta x \cdot \frac{i-1}{n} b \leq \text{Area} \leq \sum_{i=1}^{n} \Delta x \cdot \frac{i}{n} b .$$

Now the LHS is

$$\frac{ab}{n^2} \sum_{i=1}^{n} (i-1) = \frac{ab}{n^2} \sum_{i=0}^{n-1} i = \frac{ab}{n^2} \cdot \frac{n(n-1)}{2} = \frac{ab}{2}\left(1 - \frac{1}{n}\right)$$

where the first inequality is by shifting the index by 1, and the second by the formula from Lemma 10. Similarly, the RHS is

$$\frac{ab}{n^2} \sum_{i=1}^{n} i = \frac{ab}{2}\left(1 + \frac{1}{n}\right) .$$

(4) **Limit**: It follows that

$$-\frac{ab}{2n} \leq \text{Area} - \frac{ab}{2} \leq \frac{ab}{2n} .$$

Now letting $n \to \infty$ we can see that we can make the difference as close to zero as we please, so it must be exactly zero.

LEMMA 14 (Epsilon of room principle). *Let A, B be real numbers. Assume that for all $\varepsilon > 0$ (or even for arbitrarily small ε) we can show that, $-\varepsilon \leq A - B \leq \varepsilon$. Then $A = B$.*

1.2.2. The area under the graph of e^x.

PROBLEM 15. Let $a < b$. Let R be the region of the plane bounded by the x-axis, the lines $y = a$, $y = b$ and the graph of the function $f(x) = e^x$. What is the area of R?

Since $f(x) = e^x$ is monotone increasing,R contains the rectangle with base $[a, b]$ and side $[0, e^a]$ and is contained in the rectangle with base $[a, b]$ and side $\left[0, e^b\right]$. It follows that

$$(b-a)e^a \leq \text{Area}(R) \leq (b-a)e^b .$$

More generally, divide the interval into n equal subintervals, of the form $I_i = [a + ih, a + (i+1)h]$ where $h = \frac{b-a}{n}$. Let R_i be the region bounded by I_i, the lines $y = a + ih$, $y = a + (i+1)h$ and the graph of $f(x)$. Then the same reasoning shows

$$he^{a+ih} \leq \text{Area}(R_i) \leq he^{a+(i+1)h} .$$

Since $\text{Area}(R) = \sum_{0=1}^{n-1} \text{Area}(R_i)$,

$$\sum_{i=0}^{n-1} he^{a+ih} \leq \text{Area}(R) \leq \sum_{i=0}^{n-1} he^{a+(i+1)h} .$$

Taking common factors we find:

$$he^a \sum_{i=0}^{n-1} e^{ih} \leq \text{Area}(R) \leq he^a e^h \sum_{i=0}^{n-1} e^{ih} .$$

We now apply the formula
$$\sum_{i=0}^{n-1} q^i = \frac{q^n-1}{q-1}\,,$$
valid for all $q \neq 1$, to see:
$$he^a\frac{e^{nh}-1}{e^h-1} \leq \text{Area}(R) \leq he^a e^h \frac{e^{nh}-1}{e^h-1}\,.$$
Noting that $e^a(e^{nh}-1) = e^{a+\frac{b-a}{n}n} - e^a = e^b - e^a$ we rewrite this as:
$$\frac{h}{e^h-1} \leq \frac{\text{Area}(R)}{e^b-e^a} \leq \frac{h}{e^h-1}e^h\,.$$

Finally,
$$\lim_{h\to 0} e^h = e^0 = 1 \qquad \text{and} \qquad \lim_{h\to 0}\frac{e^h-1}{h} = f'(0) = 1\,.$$
Thus, given $\varepsilon > 0$ choose there is H_0 so that if $0 < h < H_0$ then $\frac{h}{e^h-1} \geq 1-\varepsilon$ and $\frac{h}{e^h-1}e^h \leq 1+\varepsilon$. Since n is arbitrary we may take $n > \frac{a-b}{H_0}$ to conclude
$$1-\varepsilon \leq \frac{\text{Area}(R)}{e^b-e^a} \leq 1+\varepsilon\,.$$
By Lemma 14 $\frac{\text{Area}(R)}{e^b-e^a} = 1$ so $\text{Area}(R) = e^b - e^a$.

1.2.3. The area under the graph of $\log^x$.

PROBLEM 16. Let $0 < a < b$. Let R be the region of the plane bounded by the x-axis, the lines $y = a$, $y = b$ and the graph of the function $f(x) = \log x$. What is the area of R?

Given $n \geq 1$ let $q = \sqrt[n]{b/a} > 1$ and divide the interval $I = [a,b]$ into the n intervals $I_i = \left[aq^i, aq^{i+1}\right]$. Let R_i be the region bounded by I_i, the lines $y = aq^i$, $y = aq^{i+1}$ and the graph of $f(x)$. Then the same reasoning again shows
$$|I_i|\log(aq^i) \leq \text{Area}(R_i) \leq |I_i|\log(aq^{i+1})\,.$$
We rewrite this as
$$aq^i(q-1)\left[\log a + i\log q\right] \leq \text{Area}(R_i) \leq aq^i(q-1)\left[\log a + (i+1)\log q\right]\,.$$

Let
$$\begin{aligned}
A = A(n) &= \sum_{i=0}^{n-1} aq^i(q-1)\log a \\
B = B(n) &= \sum_{i=0}^{n-1} aq^i(q-1)i\log q \\
C = C(n) &= \sum_{i=0}^{n-1} aq^i(q-1)\log q\,.
\end{aligned}$$
Then summing gives:
$$A + B \leq \text{Area}(R) \leq A + B + C\,.$$

We now calculate.

$$\begin{aligned}
A &= a(q-1)\log a \sum_{i=0}^{n-1} q^i \\
&= a(q-1)\log a \frac{q^n-1}{q-1} \\
&= \log a \cdot a(\frac{b}{a}-1) \\
&= (b-a)\log a\,.
\end{aligned}$$

Similarly,

$$C = (b-a)\log q\,.$$

Finally,

$$\begin{aligned}
B &= a(q-1)\log q \sum_{i=0}^{n-1} iq^i \\
&= a(q-1)\log q \left[q\frac{d}{dq}\frac{q^n-1}{q-1} \right] \\
&= a(q-1)\log q \left[\frac{nq^n}{q-1} - \frac{q(q^n-1)}{(q-1)^2} \right] \\
&= a(q-1)\frac{1}{n}\log\frac{b}{a}\frac{n(b/a)}{q-1} - a\frac{q(\frac{b}{a}-1)\log q}{q-1} \\
&= b\log\frac{b}{a} - (b-a)\frac{q\log q - 0}{q-1} \\
&\xrightarrow[q\to 1]{} b\log\frac{b}{a} - (b-a)\left[\log q + 1\right]_{q=1} \\
&= b\log\frac{b}{a} - (b-a)
\end{aligned}$$

It follows that, for any $\varepsilon > 0$,

$$(b-a)\log a + b\log\frac{b}{a} - (b-a) - \varepsilon \le \mathrm{Area}(R) \le (b-a)\log a + b\log\frac{b}{a} - (b-a) + \varepsilon\,,$$

that is

$$\mathrm{Area}(R) = (b\log b - b) - (a\log a - a)\,.$$

1.3. Definition of the integral

1.3.1. Construction (Lecture 5, 11/1/2012). Let f be a function defined and bounded on a closed interval $[a,b]$.

DEFINITION 17. A *partition* P of $[a,b]$ is a finite sequence $a = x_0 < x_1 < \cdots < x_n = b$. We say the partition has *n parts*, that the ith part has *length* $\Delta x_i = x_i - x_{i-1}$ and that the *mesh* of the partition is $\delta(P) = \max\{\Delta x_i \mid 1 \le i \le n\}$.

EXAMPLE 18. The *trivial partition* $P : a = x_0 < x_1 = b$. The *uniform partition* $x_i = a + \frac{i}{n}(b-a)$.

DEFINITION 19. Given f and P set $m_i = m_i(f;P) = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$, $M_i = M_i(f;P) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ (these exist since f is bounded!) and define the *lower* and *upper Riemann sums* by

$$\begin{aligned} L(f;P) &= \sum_{i=1}^{n} m_i \Delta x_i \,. \\ U(f;P) &= \sum_{i=1}^{n} M_i \Delta x_i \,. \end{aligned}$$

EXAMPLE 20. Let f be constant, that is $f(x) = c$ for all x. Then $m_i = M_i = c$ for all subintervals, and since $\sum_{i=1}^{n} \Delta x_i = b - a$ we have $L(f;P) = U(f;P) = c(b-a)$ for all partitions P.

DEFINITION 21. Suppose that there is a unique real number I such that for every partition P of a, b, we have $L(f;P) \le I \le U(f;P)$. We then say that f is *integrable* on $[a,b]$, say that I is the *definite integral* of f on $[a,b]$ and write

$$I = \int_a^b f(x)dx \,.$$

1.3.2. Examples and counterexamples (Lecture 5, 11/1/2012).

EXAMPLE 22. We have seen that $\int_a^b c dx = c(b-a)$.

On the other hand

EXAMPLE 23. Let $D(x) = \begin{cases} 1 & x \text{ rational} \\ 0 & x \text{ irrational} \end{cases}$ on $[0,1]$. Then $m_i = 0$, $M_i = 1$ on every interval (D takes the values $0,1$ on every interval).

More sophisticated:

EXAMPLE 24. $f(x) = \begin{cases} D(x) & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$, not integrable.

$g(x) = \begin{cases} 1 & x = 0 \\ 0 & 0 < x \le 1 \end{cases}$ which is integrable.

1.3.3. Basic Properties (Lecture 6, 13/1/2012).

PROPOSITION 25 (Linearity). *Let f, g be integrable on $[a,b]$ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is integrable and $\int_a^b (\alpha f + \beta g)dx = \alpha \int_a^b f dx + \beta \int_a^b g dx$.*

PROOF. Assume first $A, B \ge 0$ and let P be a partition so that $I(f) - \varepsilon \le L(f;P) \le I(f) \le U(f;P) \le I(f) + \varepsilon$ and also $I(g) - \varepsilon \le L(g;P) \le I(g) \le U(g;P) \le I(g) + \varepsilon$. Then

$$U(Af + Bg; P) \le A \cdot U(f;P) + B \cdot U(g;P) \le A \cdot I(f) + B \cdot I(f) + (A+B)\varepsilon$$

$$L(Af + Bg; P) \ge A \cdot L(f;P) + B \cdot L(g;P) \ge A \cdot I(f) + B \cdot I(f) - (A+B)\varepsilon \,.$$

It follows that $A \cdot I(f) + B \cdot I(g)$ is the unique number between the lower and upper sums.

It remains to consider the case of $-f$, which is done in Problem Set 2. $\square$

LEMMA 26. *Let f be integrable on $[a,b]$. If $f \ge 0$ then $\int_a^b f(x)dx \ge 0$. In particular if $f(x) \ge g(x)$ on $[a,b]$ then $\int_a^b f(x)dx \ge \int_a^b g(x)dx$.*

PROOF. In the first claim every Riemann sum is non-negative. For the second consider $h(x) = f(x) - g(x)$. $\square$

PROPOSITION 27. *Let f be integrable on $[a,b]$. Then f is integrable on every sub-interval.*

PROOF. Let $a \le a' < b' \le b$. If $P : a' = x_0 < \cdots < x_n = b'$ is a partition of $[a',b']$ then $a, x_0, \ldots, x_n, b$ is a partition of $[a,b]$ with two extra intervals. It follows that $U(f;P') - L(f;P') \le U(f;P) - L(f;P)$. $\square$

PROPOSITION 28. *Let $a < b < c$ and let f be integrable on $[a,b]$ and $[b,c]$. Then f is integrable on $[a,c]$ and*

$$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx\,.$$

PROOF. Write $I_1 = \int_a^b f(x)dx$, $I_2 = \int_b^c f(x)dx$, $I = I_1 + I_2$. Let P_1, P_2 be partitions of $[a,b], [b,c]$ respectively. Let P be their *concatenation*. Then $U(f;P) = U(f;P_1) + U(f;P_2)$ and similarly for lower sums by concatenation of finite sums. Choosing P_i so that

$$I_i - \varepsilon \le L(f;P_i) \le I_i \le U(f;P_i) \le I_i + \varepsilon\,.$$

Adding the two inequalities we have

$$I_1 + I_2 - 2\varepsilon \le L(f;P) \le I_1 + I_2 \le U(f;P) \le I_1 + I_2 + 2\varepsilon\,.$$

Taking $\varepsilon \to 0$ we are done. $\square$

EXAMPLE 29. Let $f(x) = \begin{cases} 2 & 0 \le x < 1 \\ 3 & 1 < x \le 2 \end{cases}$. Then $\int_0^2 f(x)dx = \int_0^1 2dx + \int_1^2 3dx = 5$.

DEFINITION 30. Let $a < b$ and let f be Riemann integrable on $[a,b]$. We then set $\int_b^a f(x)dx \stackrel{\text{def}}{=} -\int_a^b f(x)dx$. We also set $\int_a^a f(x)dx = 0$ for all f.

COROLLARY 31. *$\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$ holds as long as f is integrable on the intervals between a,b and b,c.*

THEOREM 32. *Let f be continuous on $[a,b]$. Then f is integrable there.*

PROOF. To be added later. $\square$

1.4. The Fundamental Theorem of Calculus (Lecture 7, 16/1/2012)

THEOREM 33. *Let $a < b$ and let f be defined and integrable on $[a,b]$. For $x \in [a,b]$ set $F(x) = \int_a^x f(t)dt$. Then:*

(1) *$F(x)$ is continuous on $[a,b]$.*
(2) *If f is continuous at $x_0 \in [a,b]$ then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

COROLLARY 34. *Let f be continuous on $[a,b]$ and let $G(x)$ be a function such that $G' = f$ on $[a,b]$. Let F be as in the Theorem. Then $F(x) = G(x) - G(a)$. In particular, $\int_a^b f(t)dt = G(b) - G(a)$.*

PROOF. Consider the function $F(x)-G(x)$ where F is as in the Theorem. It is differentiable, and $(F-G)'(x)=F'(x)-G'(x)=f(x)-f(x)=0$. It follows that $F-G$ is constant. In other words, for all x we have

$$F(x)-G(x)=F(a)-G(a)=-G(a)$$

since $F(a)=0$. It follows that

$$F(x)=G(x)-G(a)\,.$$

□

EXAMPLE 35. Note that if $G(x)=e^x$ the $\frac{dG}{dx}=e^x$, while if $G(x)=x\log x-x$ then $G'(x)=\log x$.

NOTATION 36. For a continuous function f we write $\int f(x)dx$ for a general function F so that $F'=f$. Such a function exists by the Fundamental theorem of calculus. More specifically, if F a single such function then $\int f(x)dx=F+C$ for an arbitrary constant C, since if two functions have the same derivative they differ by a constant.

Math 121: In-class worksheet for lecture 8 (17/1/2012)

EXERCISE 37. Find anti-derivatives

(1) $\int (x^3+5x^2+\sin x)dx =$

(2) $\int e^{5x}dx =$

(3) $\int xe^{x^2}dx =$

(4) $\int \sqrt{x+5} =$

EXERCISE 38. Find:

(1) Let $F(x) = \int_{x^2}^{\cos x}\left(e^{e^t} - \tan t\right)dt$. Find $F'(x) =$

(2) (2010 final) Let $G(x) = \frac{d}{dx}\left[x^2\int_0^{x^2}\frac{\sin u}{u}du\right] - 2x\int_0^{x^2}\frac{\sin u}{u}du$. Find $G\left(\sqrt{\frac{\pi}{2}}\right)$

EXERCISE 39. For which $a<b$ is $\int_a^b\left(4x-x^2\right)dx$ largest?

1.5. Numerical integration

The midpoint rule is considered in problem set 7.

1.6. Improper integrals

1.6.1. Open interval.

PROBLEM 40. Define $f(x)$ on $[0,1]$ by $f(x) = \begin{cases} x^{-1/2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then f is not Riemann integrable on $[0,1]$, but at least formally we have $\int f(x)dx = 2\sqrt{x} + C$ so we ought to have $\int_0^1 f(x)dx = 2$.

Assume that f is only defined on the half-open interval $(a,b]$, and assume that f is Riemann integrable on every closed interval $[a+\varepsilon, b]$ where $\varepsilon > 0$. Suppose that $\lim_{y\to a} \int_y^b f(x)dx$ exists. In that case we say that the *improper integral* $\int_a^b f(x)dx$ exists and set

$$\int_a^b f(x)dx = \lim_{y\to a} \int_y^b f(x)dx\,.$$

1.7. Appendix: The Real numbers

1.7.1. Elementary properties of fields.

DEFINITION 41. An *field* is a quintuple $(F, +, \cdot, 0, 1)$ satisfying:

(1) Language: F is a set, $+ \colon F\times F \to F$ ("addition") and $\cdot \colon F \times F \to F$ ("multiplication") are binary operations; $0, 1 \in F$ are distinct elements ("zero", "one").
(2) Field axioms
  (a) Addition: For all $x,y,z \in F$ we have:
    - $(x+y)+z = x+(y+z)$ ("associative law")
    - $x+0 = x$ ("zero")
    - There is $x' \in F$ such that $x+x' = 0$ ("negation")
    - $x+y = y+x$ ("commutative law")
  (b) Multiplication: For all $x,y,z \in F$ we have:
    - $(x\cdot y)\cdot z = x\cdot(y\cdot z)$ ("associative law")
    - $x \cdot 1 = x$ ("identity")
    - If $x \neq 0$ there is y such that $x\cdot y = 1$ ("inverse")
    - $x\cdot y = y \cdot x$ ("commutative law")
  (c) Distributive law: $x\cdot(y+z) = x\cdot y + x \cdot z$.

DEFINITION 42. An ordered field is a sextuple $(F, +, \cdot, 0, 1, <)$ where:

(1) $(F, +, \cdot, 0, 1)$ is a field.
(2) Order:$<$ is a binary relation ("less than"), and for all $x,y,z \in F$
  - If $x < y$ and $y < z$ then $x < z$ ("transitivity").
  - Exactly one of $x < y$, $x = y$, $y < x$ holds ("trichotomy").
(3) Compatibility with field operations: For all $x,y,z \in F$
  - If $x < y$ then $x+z < y+z$.
  - If $x < y$ and $z > 0$ then $x \cdot z < y \cdot z$.

EXAMPLE 43. $(\mathbb{Q},+,\cdot,0,1,<)$, $(\mathbb{R},+,\cdot,0,1,<)$

LEMMA 44. *Let F be a field, and let $a,b,c \in F$ with $a \neq 0$. Then the equation $ax+b=c$ has the unique solution $x = a'(c+b')$, where $a^{'}$ is such that $aa' = a'a = 1$ and b' is such that $b+b' = b'+b = 0$..*

PROOF. If $ax+b=c$ then $ax = ax+0 = ax+(b+b') = (ax+b)+b' = c+b'$. It follows that $x = 1x = (a'a)x = a'(ax) = a'(c+b')$. Conversely, if $a\cdot a^{-1} = 1$ and $b+b'=0$ then $a\cdot(a^{-1}(b'+c))+b = (a\cdot a^{-1})(b'+c)+b = 1\cdot(b'+c)+b = (b'+c)+b = c+(b'+b) = c+0 = 0$. □

COROLLARY 45. *For all $a \in F$:*

(1) *The equation $1x+a=a$ has a unique solution; it follows that zero is unique.*
(2) *If $a \neq 0$, the equation $ax=a$ has the unique solution; it follows that the identity is unique.*
(3) *The equation $a+x=0$ has a unique solution; we denote it $-a$.*
(4) *if $a \neq 0$, the equation $ax=1$ has a unique solution, to be denoted a^{-1}.*

LEMMA 46. *Let F be a field. Then:*

(1) *For all $x \in F$, we have $x\cdot 0 = 0$.*
(2) *For all $x \in F$, we have $x\cdot(-1) = -x$.*

PROOF.

(1) We have $0 = 0+0$ so by the zero and distributive laws $x\cdot 0 = x\cdot(0+0) = x\cdot 0 + x\cdot 0$. There is y such that $x\cdot 0+y=0$ so adding y to both sides we get
$$0 = x\cdot 0+y = (x\cdot 0+x\cdot 0)+y = x\cdot 0+(x\cdot 0+y) = x\cdot 0+0 = x\cdot 0\,.$$
(2) $x+\cdot(-x) = 0 = x\cdot 0 = x\cdot(1+(-1)) = x\cdot 1+x\cdot(-1)$.

□

PROPOSITION 47. *(Generalized associative law)*

PROPOSITION 48. *(Generalized commutative law)*

DEFINITION 49. (Powers) Let $a \in F$. Set $a^0 = 1$ and for a natural number n set $a^{n+1} = a^n \cdot a$. If $a \neq 0$ then for a negative integer n set $a^n \stackrel{\text{def}}{=} (a^{-1})^{-n}$.

PROPOSITION 50. *(Power laws) For all $a,b \in F$ and $n,m \in \mathbb{N}$ we have*
$$\begin{aligned} a^n\cdot a^m &= a^{n+m} \\ (a^n)^m &= a^{nm} \\ a^n b^n &= (ab)^n\,. \end{aligned}$$
If a,b are non-zero then these laws hold for all $n,m \in \mathbb{Z}$.

1.7.2. The order.

LEMMA 51. *Let F be an ordered field and let $x \in F$ be non-zero. Then exactly one of $x,-x$ is positive.*

PROOF. If $x > 0$ then $x+(-x) > 0+(-x)$. If $x < 0$ then $x+(-x) < 0+(-x)$. □

COROLLARY 52. *For all $x \in F$ we have $x^2 \geq 0$ with equality iff $x = 0$.*

PROOF. If $x > 0$ then $x \cdot x > 0 \cdot x = 0$. If $x = 0$ then $x^2 = 0$. If $x < 0$ then $-x > 0$ so $x^2 = (-1)(-1)x \cdot x = (-x)^2 > 0$. $\square$

COROLLARY 53. $1 > 0$.

PROOF. $1 = 1^2$. $\square$

DEFINITION 54. If $x \in F$ the *absolute value* of x is the element of F given by $|x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases}$.

LEMMA 55. *(Norm) For all $x, y \in F$:*

(1) $|x| \geq 0$ *and* $|x| = 0$ *iff* $x = 0$.
(2) *(Triangle inequality)* $|x+y| \leq |x| + |y|$, $|x-y| \geq |x| - |y|$.
(3) $|xy| = |x|\,|y|$.

PROOF. Divide into cases. $\square$

1.7.3. Completeness. The following formalizes our idea that there are "no holes" in the real number line.

AXIOM 56 (Completeness of $\mathbb{R}$). *Let A, B be non-empty sets of real numbers such that:*

(1) *For every $a \in A$, $b \in B$ we have $a \leq b$.*
(2) *For every $\varepsilon > 0$ there are $a \in A$, $b \in B$ with $b - a \leq \varepsilon$.*

Then there is a real number L such that $a \leq L \leq b$ for all $a \in A$, $b \in B$.

LEMMA 57. *The number L in the axiom is unique.*

PROOF. Let A, B be as in the axiom, and let L, L' satisfy the conclusion. Given $\varepsilon > 0$ let $a \in A$, $b \in B$ satisfy $b \leq a + \varepsilon$. Then $L, L' \in [a, a+\varepsilon]$ so $|L - L'| \leq \varepsilon$. $\square$

Lower and upper bounds.

DEFINITION 58. Let F be an ordered field and let $A \subset F$ be non-empty. Call $M \in F$ an *upper bound* for A if for all $x \in A$ we have $x \leq M$. Call $m \in F$ a *lower bound* for A if for all $x \in A$ we have $x \geq m$.

Call A *bounded above* if it has an upper bound, *bounded below* if it has a lower bound, *bounded* if both hold, and *unbounded* if it is not bounded.

LEMMA 59. *A non-empty $A \subset \mathbb{R}$ is bounded iff there is $M \in \mathbb{R}$ such that for all $x \in A$, $|x| \leq M$.*

NOTATION 60. Let $a < b$ be real numbers. We write

$$\begin{aligned}
(a,b) &= \{x \in \mathbb{R} \mid a < x < b\} && \text{("open interval")} \\
[a,b) &= \{x \in \mathbb{R} \mid a \leq x < b\} \\
(a,b] &= \{x \in \mathbb{R} \mid a < x \leq b\} \\
[a,b] &= \{x \in \mathbb{R} \mid a \leq x \leq b\} && \text{("closed interval")}
\end{aligned}$$

and also

$$\begin{array}{rcll}
(a,\infty) & = & \{x\in\mathbb{R} \mid a<x\} & \text{("open ray")} \\
[a,\infty) & = & \{x\in\mathbb{R} \mid a\le x\} & \text{("closed ray")} \\
(-\infty,b) & = & \{x\in\mathbb{R} \mid x<b\} & \text{("open ray")} \\
(-\infty,b] & = & \{x\in\mathbb{R} \mid x\le b\} & \text{("closed ray")}
\end{array}$$

EXAMPLE 61. $[0,1]$ is bounded. $(-\infty,1)$ is bounded above while $(1,\infty)$ is not bounded above but bounded below.

LEMMA 62. *A non-empty $A\subset\mathbb{R}$ is bounded iff it is contained in an interval.*

REMARK 63. Note that it does not matter if the interval is open or closed. Why?

EXAMPLE 64. Formulate a condition for one-sided boundedness using rays.

The least upper bound property. Fix a non-empty set $A\subset\mathbb{R}$ which is bounded above.

LEMMA 65. *Let $M<M'$ be two real numbers. If M is an upper bound for A then so is M'. The set $\{M\in\mathbb{R} \mid M \text{ is an upper bound for } A\}$ is bounded below.*

PROOF. Exercise. □

DEFINITION 66. Say that $M\in\mathbb{R}$ is a *least upper bound* of A (l.u.b. for short) if
(1) M is an upper bound on A.
(2) For every upper bound M', $M'\ge M$.

LEMMA 67. *If A has a least upper bound then it has exactly one. In that case we write* $\sup A$ *(the "supremum" of A) for its least upper bound.*

LEMMA 68. *If $M\in A$ is an upper bound for A then A is the least upper bound of A. In that case we call M the* maximum *of A and write $M=\max A$.*

THEOREM 69. *There exists $x\in\mathbb{R}$ such that $x^2=2$.*

1.7.4. Induction. Call $I\subset\mathbb{R}$ *inductive* if $0\in I$ and if whenever $x\in I$ we also have $x+1\in I$.

EXAMPLE 70. $\mathbb{R}$ is inductive; $\mathbb{R}_{\ge 0}$ is inductive. $\mathbb{R}_{\ge 1}$ is not (does not contain zero), and neither is $\{0,1\}$.

PROPOSITION 71. *(Archimedean property) Every inductive set is not bounded above. Equivalently, if $I\subset\mathbb{R}$ is inductive and $M\in\mathbb{R}$ then there is $n\in I$ such that $n>M$.*

DEFINITION 72. $\mathbb{N}=\bigcap\{I\subset\mathbb{R} \mid I \text{ inductive}\}\subset\mathbb{R}$.

PROPOSITION 73. $\mathbb{N}$ *is inductive.*

COROLLARY 74. *(Proof by induction) Let $A\subset\mathbb{N}$ be inductive. Then $A=\mathbb{N}$.*

PROOF. $\mathbb{N}\subset A$ holds be definition. □

COROLLARY 75. *(Archimedean property) Let $x\in\mathbb{R}$. Then there is $n\in\mathbb{N}$ such that $n>x$. It also follows that if $\varepsilon>0$ then there is $n\in\mathbb{N}$such that $\frac{1}{n}>\varepsilon$.*

The following is not part of the material:

THEOREM 76. *(Strong induction) Let $A \subset \mathbb{N}$ have the property that for all $n \in \mathbb{N}$.*

LEMMA 77. *(Discreteness) There is no integer b satisfying $0 < b < 1$.*

COROLLARY 78. *For any integer n there is no integer a satisfying $n < a < n+1$.*

THEOREM 79. *$\mathbb{N}$ is closed under addition and multiplication: if $x, y \in \mathbb{N}$ then so are $x+y$ and $x \cdot y$.*

CHAPTER 2

Techniques of integration

Basic idea: laws for derivatives induce corresponding laws for anti-derivatives.

2.1. Integration by substitution (Lecture 9, 18/1/2012)

- Chain rule: $\frac{d}{dx}f(g(x)) = \frac{d}{du}f(u) \restriction_{u=g(x)} \cdot \frac{dg}{dx}$.
- So: $\int f'(g(x))g(x)dx = f(g(x)) + C$.
- Useful way to think about this: try to break up integrand into a function $f(u)$ and a derivative $\frac{du}{dx}dx$.

EXAMPLE 80. $\int \cos^2 x\,\mathrm{d}x$.

- Idea: half-angle formula says $\cos(2x) = 2\cos^2 x - 1$ so $\cos^2 x = \frac{1}{2}(1+\cos(2x))$, and

$$\begin{aligned}\int \cos^2 x\,\mathrm{d}x &= \frac{1}{2}\int 1\,\mathrm{d}x + \frac{1}{2}\int \cos(2x)\,\mathrm{d}x \\ &= \frac{x}{2} + \frac{1}{4}\int \cos(2x)\,\mathrm{d}(2x) \\ &= \frac{x}{2} + \frac{1}{4}\sin(2x)\,.\end{aligned}$$

2.2. Integration by parts (Lecture 10, 20/1/2012)

- Product rule: $\frac{d}{dx}(fg) = f\frac{dg}{dx} + \frac{df}{dx}g$.
- So: $\int f'g dx = fg - \int fg' dx$.
  - In words: if we are integrating a product we can replace one factor by the integral and the other factor by its derivative at the cost of a minus sign and a factor fg.
  - Useful if the passage from f' (known) to f is not hard, while going from g to g' simplifies the problem.
- Definite integral:

$$\int_a^b f'g dx = [fg]_{x=a}^{x=b} - \int_a^b fg' dx$$

  where $[F]_{x=a}^{x=b} \stackrel{\text{def}}{=} F(b) - F(a)$ for any function F.

Math 121: In-class worksheet for lecture 10

EXERCISE 81. Find anti-derivatives

(1) $\int xe^x dx =$

(2) $\int x^2 e^x =$

(3) $I_n = \int x^n e^x dx$ in terms of I_{n-1},

(4) $\int \cos^2 x dx.$

(5) $\int \arcsin x\, dx$

EXERCISE 82. Find:

(1) $\int \frac{1}{5-x} dx =$
(2) $\int \frac{1}{1+x^2} dx =$
(3) $\int \frac{x}{1+x^2} dx =$
(4) $\int \frac{1}{1-x^2} dx$=

EXAMPLE 83. $\int \log x \,\mathrm{d}x = \int 1 \cdot \log x \,\mathrm{d}x = x \log x - \int x \cdot \frac{1}{x} dx = x \log x - x + C$.

2.3. Rational fractions: the partial fractions expansion (Lectures 11-13, 23-25/1/12)

2.3.1. Lecture 11: examples from end of last worksheet.

EXAMPLE 84. Basic examples for this lecture:

(1) $\int \frac{\mathrm{d}x}{5-x} = \log|x-5| + C$.
(2) $\int \frac{\mathrm{d}x}{1+x^2} = \arctan x + C$
(3) $\int \frac{x}{1+x^2}\mathrm{d}x$: note that if $u = 1+x^2$ then $\mathrm{d}u = 2x\,\mathrm{d}x$. Changing variables this way we have $\int \frac{x\,\mathrm{d}x}{1+x^2} = \frac{1}{2}\int \frac{\mathrm{d}u}{u} = \frac{1}{2}\log|u| = \frac{1}{2}\log(1+x^2)$.
(4) $\int \frac{\mathrm{d}x}{1+x+x^2}$. Not so obvious what to do. Idea: complete the square – rewrite $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{4} = \frac{3}{4} + \left(x+\frac{1}{2}\right)^2$. We now have the idea of working in terms of the variable $u = x + \frac{1}{2}$ (i.e. shifting the axis). Then $\int \frac{\mathrm{d}x}{1+x+x^2} = \int \frac{\mathrm{d}u}{\frac{3}{4}+u^2}$. This is similar to 2 but not quite.

    We can get there by taking a common factor of $\frac{3}{4}$ to get:
$$\int \frac{\mathrm{d}u}{\frac{3}{4}+u^2} = \frac{4}{3}\int \frac{du}{1+\frac{4}{3}u^2} = \frac{4}{3}\int \frac{du}{1+\left(\frac{2u}{\sqrt{3}}\right)^2}.$$
    The final idea is to rescale the variable – change to $v = \frac{2u}{\sqrt{3}}$.

REMARK 85. On completing the square.

EXAMPLE 86. $\int \frac{\mathrm{d}x}{1-x^2}$. Trick: $\frac{1}{1-x^2} = \frac{1}{2}\left[\frac{1}{1-x} + \frac{1}{1+x}\right]$. Conclude that
$$\int \frac{\mathrm{d}x}{1-x^2} = \frac{1}{2}\left[-\log|1-x| + \log|1+x|\right] = \frac{1}{2}\log\left|\frac{1+x}{1-x}\right|.$$
This can actually be discovered systematically:

- Plot $\frac{1}{1-x^2}$. Note the “bad points” $x = \pm 1$ which seem to appear in the formula above.
- Near $x = 1$, $\frac{1}{1-x^2} = \frac{1}{1-x}\frac{1}{1+x}$ behaves roughly like $\frac{1}{2(1-x)}$. In other words, when x is close to 1 $\frac{1}{1-x^2}$ blows up like $\frac{1}{2\cdot(\text{distance to }1)}$.
- Subtract $\frac{1}{1-x^2} - \frac{1/2}{1-x}$ get $\frac{1/2}{1+x}$.

REMARK 87. Note the idea of *asymptotics:* we ask not just where $\frac{1}{1-x^2}$ blows up, but *how fast* it blows up there.

EXAMPLE 88. $\frac{1}{x^3-x}$.

(1) Factor denominator: $x^3 - x = x(x-1)(x+1)$.
(2) Bad points: $0, 1, -1$.
(3) Asymptotics:
    (a) Near $x = 0$ $\frac{1}{x(x-1)(x+1)} \approx -\frac{1}{x}$.
    (b) Near $x = 1$, $\frac{1}{x(x-1)(x+1)} \approx \frac{1}{2(x-1)}$
    (c) Near $x = -1$, $\frac{1}{x(x-1)(x+1)} \approx -\frac{1}{2(x+1)}$.

(4) Try this out:

$$\begin{aligned}
-\frac{1}{x}+\frac{1}{2(x-1)}-\frac{1}{2(x+1)} &= \frac{-2(x^2-1)+x(x+1)-x(x-1)}{2x(x+1)(x-1)} \\
&= \frac{2-2x^2+2x^2+x-x}{2(x^3-x)} \\
&= \frac{1}{x^3-x}.
\end{aligned}$$

QUESTION 89. *What about* $\frac{1}{x^3-1}$*?*

(1) *Factor denominator:* $x^3-1=(x-1)(x^2+x+1)$.
(2) *Bad point:* 1.
(3) *Asymptotics:* $\frac{1}{(x-1)(x^2+x+1)} \sim \frac{1}{3(x-1)}$.
(4) *Subtract:* $\frac{1}{x^3-1}-\frac{1}{3(x-1)}=\frac{3-(x^2+x+1)}{3(x-1)(x^2+x+1)}=-\frac{x^2+x-2}{3(x-1)(x^2+x+1)}$. *Note that* x^2+x-2 *has a root at* $x=1$ *(generalization in the next lecture). Cancel factor of* $x-1$ *we are left with:*

$$\frac{1}{x^3-1}=\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}.$$

EXAMPLE 90. $\int \frac{dx}{x^3-1}=\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x+2}{x^2+x+1}\,dx$. The first integral is elementary. For the second:

- First idea: By Example 84(4) from start of lecture it is useful to work in terms of the variable $u=x+\frac{1}{2}$. Get: $\int\frac{x+2}{x^2+x+1}\,dx=\int\frac{u+\frac{3}{2}}{\frac{3}{4}+u^2}\,du$.
- Second idea: split integrand into two summands: $\int\frac{u+\frac{3}{2}}{\frac{3}{4}+u^2}\,du=\int\frac{u}{\frac{3}{4}+u^2}\,du+\frac{3}{2}\int\frac{1}{\frac{3}{4}+u^2}\,du$. The second of those is dealt with in Example 84(4) above.
- Third idea: $u\,du=\frac{1}{2}\,d(\frac{3}{4}+u^2)$ so $\int\frac{u}{\frac{3}{4}+u^2}\,du=\frac{1}{2}\log\left(\frac{3}{4}+u^2\right)$.

REMARK 91. This can be generalized to $\int\frac{Dx+E}{Ax^2+Bx+C}\,dx$ where the denominator is irreducible, following the steps:

(1) Complete the square in the denominator.
(2) Shift to the variable $u=x+\frac{B}{2A}$, in which the denominator is $A\left(u^2+\frac{4AC-B^2}{4A^2}\right)$ and the numerator is $Du+(E-\frac{DB}{2A})$.
(3) Consider $\frac{D}{A}\int\frac{u\,du}{u^2+\frac{4AC-B^2}{4A^2}}$ and $\left(E-\frac{DB}{2A}\right)\int\frac{du}{u^2+\frac{4AC-B^2}{4A^2}}$ separately. The first is $\frac{D}{2A}\log\left(u^2+\frac{4AC-B^2}{4A^2}\right)$ and the second is $\left(E-\frac{DB}{2A}\right)\sqrt{\frac{4A^2}{4AC-B^2}}\arctan\left(\sqrt{\frac{4A^2}{4AC-B^2}}u\right)$ by changing variables via $v=\sqrt{\frac{4A^2}{4AC-B^2}}u$.

2.3.2. Lecture 12-13: General scheme.

EXAMPLE 92. $\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$. Subtracting $\frac{1}{3(x-1)}$ get $\frac{x^2+x-2}{x^3-1}$. Know numerator vanishes at 1 so divisible by $x-1$; cancelling get $\frac{x+2}{x^2+x+1}=\frac{(x+\frac{1}{2})}{\frac{3}{4}+(x+\frac{1}{2})^2}+\frac{3}{2}\frac{1}{\frac{3}{4}+(x+\frac{1}{2})^2}$.

$\frac{1}{x^3-x^2} = \frac{1}{x^2(x-1)}$. Near 1 $\sim \frac{1}{x-1}$. Near 0 $\sim -\frac{1}{x^2}$. Subtracting, we are left with

$$\frac{1}{x^3-x^2} - \frac{1}{x-1} + \frac{1}{x^2} = \frac{1-x^2+x-1}{x^2(x-1)} = \frac{x-x^2}{x^2(x-1)} = \frac{1}{x}$$

so

$$\frac{1}{x^3-x^2} = \frac{1}{x-1} - \frac{1}{x^2} + \frac{1}{x}.$$

NOTE 93. At each "bad point" we found the *worst* blowup. We then subtracted that. What's left blows up, but *more slowly*.

EXAMPLE 94. $\frac{1}{x^2(x-1)^2}$. There will be a $\frac{C}{x^2}$, $\frac{D}{(x-1)^2}$. After taking those out left with $\frac{E}{x}$, $\frac{F}{x-1}$.

DEFINITION 95. Let f,g be functions, $a \in \mathbb{R}$. We say that f,g are *asymptotic to each other near a* and write $f \sim_a g$ *if* $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$.

EXAMPLE 96. If f is continuous and non-vanishing at a then $f \sim_a f(a)$. Also $\sin x \sim_0 x$ and $\frac{1}{\sin x} \sim_a \frac{1}{x}$.

EXERCISE 97. This is an equivalence relation: If $f \sim_a f$, if $f \sim_a g$ then $g \sim_a f$ and if $f \sim_a g$ and $g \sim_a h$ then $f \sim_a h$.

DEFINITION 98. f has a *zero of order/multiplicity m at a* if $f \sim_a C(x-a)^m$ *where* $C \neq 0$. f *has a* pole *if* $f \sim_a \frac{C}{(x-a)^m}$.

THEOREM 99. *(Division with remainder) Let $P,Q \in \mathbb{R}[x]$ be polynomials. Then there exist unique polynomials A,B so that $P = AQ + B$ and such that* $\deg B < \deg Q$.

LEMMA 100. *$P,Q \in \mathbb{R}[x]$ be polynomials with no common factors. Suppose that $(x-a)^m$ exactly divides Q, where $m \geq 1$, so that $Q(x) = (x-a)^m R(x)$. Then*

$$\frac{P}{Q} \sim_a \frac{P(a)}{R(a)} \frac{1}{(x-a)^m}$$

and

$$\frac{P}{Q} - \frac{P(a)/R(a)}{(x-a)^m} = \frac{\tilde{P}}{\tilde{Q}}$$

where $\tilde{Q} = (x-a)^{\tilde{m}} R(x)$ with $\tilde{m} < m$, and $\tilde{P}$, $\tilde{Q}$ have no common factors.

PROOF. Check $P(a) \neq 0$ and then the first claim is clear.

$$\frac{P}{Q} - \frac{P(a)/R(a)}{(x-a)^m} = \frac{1}{R(x)(x-a)^m}\left[P(x) - \frac{P(a)}{R(a)}R(x)\right].$$

Now the polynomial in the parenthesis vanishes at $x = a$ so it is of the form $(x-a)^t \tilde{P}(x)$ where $t \geq 1$ and $\tilde{P}(a) \neq 0$. Cancelling a factor $(x-a)^{\min\{m,t\}}$ gives $\tilde{m} = m - \min\{m,t\} < m$. Now consider an irreducible common factor T of $\tilde{P},\tilde{Q}$. Either T divides R or $\tilde{m} \geq 1$ and T divides $(x-a)^{\tilde{m}}$. In the first case, T would divide $P(x) - \frac{P(a)}{R(a)}R(x)$ (a multiple of $\tilde{P}$) and and R hence also P. It would also divide Q (a multiple of $\tilde{Q}$) a contradiction. In the second case this would mean $\tilde{P}(a) = 0$ which is impossible – in this case $t < m$. $\square$

PROPOSITION 101. *Let $P,Q \in \mathbb{R}[x]$ with $Q \neq 0$. Let $\{a_i\}_{i=1}^r$ be the zeroes of g, of multiplicity m_i. Then $\frac{P}{Q} = \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{C_{i,j}}{(x-a_i)^j} + \frac{\tilde{P}}{\tilde{Q}}$ where $\tilde{P},\tilde{Q} \in \mathbb{R}[x]$ with $\tilde{Q}$ non-vanishing.*

PROOF. By induction on the degree of Q, applying Lemma 100. □

THEOREM 102. *(Expansion in Partial Fractions) Let $P,Q \in \mathbb{R}[x]$ with $Q \neq 0$. Then there are $\{a_i\}_{i=1}^r \subset \mathbb{R}$, irreducible quadratics $\{T_k\}_{k=1}^s$, integers m_i and n_k and constants C_{ij} D_{kl}, and a polynomials R so that*

$$\frac{P(x)}{Q(x)} = \sum_{i=1}^r \sum_{j=1}^{m_i} \frac{C_{ij}}{(x-a_i)^j} + \sum_{k=1}^s \sum_{l=1}^{n_k} \frac{D_{kl}}{T_k^l} + R(x).$$

- Quadratic factors: say $\frac{P}{Q} = \frac{P}{R \cdot T^m}$ where T is quadratic irreducible. Again there is C so that $\frac{P}{Q} - \frac{C}{T^m} = \frac{\tilde{P}}{R \cdot T^{\tilde{m}}}$ with $\tilde{m}$ smaller. More difficult to express the C.
- For this course: there will be at most one quadratic factor, to be found by subtracting the other parts.

2.4. Substitution II (27-30/1/2012)

Recall:

$$\int f(g(x))g'(x)dx = \int f(u)du$$

where $u = g(x)$. Now exchange the roles of x, u:

EXAMPLE 103 (Area of the disc). $\int_{-1}^1 \sqrt{1-x^2}\,\mathrm{d}x$. Try $x = \sin\theta$, so $\mathrm{d}x = \cos\theta\,\mathrm{d}\theta$. What about endpoints?

EXAMPLE 104. (Half-angle) Consider $\int \frac{\mathrm{d}\theta}{\sin\theta + \cos\theta}$.

2.5. Improper integrals and comparison (31/1/2012-3/2/2012)

DEFINITION 105. Let f be defined on $[a,\infty)$ and Riemann integrable on every interval $[a,b]$ where $b \geq a$. If the limit $\lim_{T\to\infty} \int_a^T f(x)\,\mathrm{d}x$ exists we say that the "improper integral$\int_a^\infty f(x)\,\mathrm{d}x$ converges" and write

$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{T\to\infty} \int_a^T f(x)\,\mathrm{d}x.$$

If the limit does not exist we say the integral *diverges*.

EXAMPLE 106. $\int_a^T e^{-x}\,\mathrm{d}x = e^{-a} - e^{-T} \xrightarrow[T\to\infty]{} e^{-a}$. For $a > 0$ we have and $p \neq 1$ we have

$$\int_a^T \frac{\mathrm{d}x}{x^p} = \left[\frac{x^{1-p}}{1-p}\right]_{x=a}^{x=T} = \frac{a^{1-p}}{p-1} - \frac{T^{1-p}}{p-1} \xrightarrow[T\to\infty]{} \begin{cases} \frac{a^{1-p}}{p-1} & p > 1 \\ \infty & p < 1 \end{cases}.$$

Also, $\int_a^T \frac{\mathrm{d}x}{x} = \log\frac{T}{a} \xrightarrow[T\to\infty]{} \infty$.

- Idea: convergence of the integral is most often a question about the *rapid decay* of the integrand.
- Caution: sometimes can have congergence with slow decay due to cancellation.

EXAMPLE 107. $\lim_{T\to\infty} \int_a^T \frac{\sin x}{x^p}\,\mathrm{d}x$ exists for all $p > 0$ (to be seen later).

DEFINITION 108. Similarly, if f is bounded on $(-\infty, a]$ and Riemann integrable on every bounded subinterval we set

$$\int_{-\infty}^{a} f(x)\,\mathrm{d}x = \lim_{T\to-\infty} \int_{T}^{a} f(x)\,\mathrm{d}x\,.$$

If f is bounded on the entire axis we say $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ converges if $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_{-\infty}^a f(x)\,\mathrm{d}x$ exist separately, and set $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^a f(x)\,\mathrm{d}x + \int_a^\infty f(x)\,\mathrm{d}x$

EXERCISE 109. The last definition is independent of the choice of a. Compare $\int_{-T}^{T} x\,\mathrm{d}x$, $\int_{-T}^{T^2} x\,\mathrm{d}x$ and $\int_{-T^2}^{T} x\,\mathrm{d}x$.

REMARK 110.

(1) Note that the notation $\int_a^\infty f(x)\,\mathrm{d}x$ does not directly correspond to a calculation with Riemann sums. It is a shorthand for a limit of definite integrals.
(2) Suppose that f is continuous on $[a,\infty)$ with anti-derivative $F(x)$. Then $\int_a^T f(x)\,\mathrm{d}x = F(T) - F(a)$ so the integral exists if and only if $\lim_{T\to\infty} F(T)$ exists.

PROPOSITION 111. *Suppose $\int_a^\infty f(x)\,\mathrm{d}x$ and $\int_a^\infty g(x)\,\mathrm{d}x$ converge. Then for any α,β $\int_a^\infty (\alpha f + \beta g)\,\mathrm{d}x$ converges and $\int_a^\infty (\alpha f + \beta g)\,\mathrm{d}x = \alpha \int_a^\infty f\,\mathrm{d}x + \beta \int_a^\infty g\,\mathrm{d}x$.*

PROOF. Linearity of integrals and limits. □

DEFINITION 112. Suppose instead that f is defined on $[a,b)$ and Riemann integrable on every interval $[a,T]$ where $T < b$. If the limit $\lim_{T\to b} \int_a^T f(x)\,\mathrm{d}x$ exists we say that the "improper integral $\int_a^b f(x)\,\mathrm{d}x$ converges" and write $\int_a^b f(x)\,\mathrm{d}x$ for the value of the limit. A similar definition is made for the lower endpoint.

2.5.1. Asymptotics I – positive integrands. Suppose $f(x) \geq 0$ for all $x \geq a$. Then we think of $\int_a^\infty f(x)\,\mathrm{d}x$ as the area of an unboudned region. It's clear the area of the region is at least $\int_a^T f(x)\,\mathrm{d}x$ for all $T \geq a$. Also, $\int_a^T f(x)\,\mathrm{d}x$ is increasing in T.

THEOREM 113. *Let $F(T)$ be an increasing function of T. Then either $F(T)$ is bounded and $\lim_{T\to\infty} F(T)$ exists or F is unboudned and $\lim_{T\to\infty} F(T) = \infty$.*

NOTATION 114. In this case we write $\int_a^\infty f(x)\,\mathrm{d}x < \infty$ to indicate convergence.

COROLLARY 115. *$\int_a^\infty f\,\mathrm{d}x$ converges if and only if all the integrals $\int_a^T f\,\mathrm{d}x$ are uniformly bounded. In particular, suppose that $0 \leq f(x) \leq g(x)$ for all x. Then:*

(1) *If $\int_a^\infty g\,\mathrm{d}x < \infty$ then $\int_a^\infty f\,\mathrm{d}x < \infty$.*
(2) *If $\int_a^\infty f\,\mathrm{d}x = \infty$ then $\int_a^\infty g\,\mathrm{d}x = \infty$*

COROLLARY 116. *Suppose that f,g are non-negative and that $0 < A \leq \frac{f(x)}{g(x)} \leq B$ for all x large enough. Then $\int_a^\infty f(x)\,\mathrm{d}x, \int_a^\infty g(x)\,\mathrm{d}x$ either both converge or both diverge.*

2.5.2. Absolute convergence.

DEFINITION 117. Say $\int_a^\infty f(x)\,\mathrm{d}x$ (or $\int_a^b f(x)\,\mathrm{d}x$) *converges absolutely* if the same integral with $|f(x)|$ instead converges.

THEOREM 118. *Suppose $\int_a^\infty f(x)\,\mathrm{d}x$ converges absolutely. Then the integral converges and*

CHAPTER 3

Applications

Paradigm:

(1) *Parametrize*: choose axes, co-ordinates etc.
(2) *Slice*: Divide the quantity to be calculated into infinitesimal pieces labelled by the parameter.
(3) *Integrate:* Write the quantity to be calculated as an integral over the slices and evaluate the integral.

3.1. Volume

3.1.1. Slicing by example (6/2/2012).

- "Infinitesimal approach" – cut up volume to be computed into infinitely many infinitely small pieces; add up contributions using integral.

EXAMPLE 119. The volume of the ball

(1) Parametrize $B = \left\{(x,y,z) \mid x^2+y^2+z^2 \leq R^2\right\}$. Decide to slice perpendicular to z axis.
(2) Partition interval of z values: $-R = z_0 < z_1 < \cdots < z_n = R$.
(3) To each part associate a "slab" $\left\{(x,y,z) \mid x^2+y^2+z^2 \leq R^2,\ z \in [z_{i-1}, z_i]\right\}$. This is:
    (a) Approximately cylindrical, of small height $\Delta z_i = z_i - z_{i-1}$.
    (b) Approximately cylindrical, so of volume about $A(z_i)\Delta z_i$ where $A(z_i)$ is the cross-sectional area of the cylinder.
(4) Conclude that the volume of the ball is about $\sum_{i=1}^{n} A(z_i)\Delta z_i$.
    (a) This looks like a Riemann sum!
    (b) Not an upper or lower sum, but bounded by them.
    (c) We will dispense with the formal construction of outer and inner cylinders and simply say:
    $$\mathrm{vol}(B) = \int_{z=-R}^{z=R} A(z)\,\mathrm{d}z$$
    where $A(z)$ is the cross-sectional area at z.
(5) Slice at z is the disc $\left\{(x,y) \mid x^2+y^2 \leq R^2 - z^2\right\}$ of area $\pi\left(R^2 - z^2\right)$.
(6) Volume is therefore
$$\int_{-R}^{R} \pi\left(R^2 - z^2\right)\mathrm{d}z = \frac{4\pi}{3}R^3$$
.

EXAMPLE 120 ("Improper volume"). Let R be the "hill" $R = \left\{(x,y,z) \mid 0 \leq z \leq e^{-x^2-y^2}\right\}$. Slice perpendicular to x axis.

(1) Slice at x looks like $\left\{(y,z) \mid 0 \le z \le e^{-x^2-y^2}\right\}$ so its area is the area under the graph of $z(y) = e^{-x^2}e^{-y^2}$. It follows that

$$A(x) = \int_{y=-\infty}^{y=+\infty} e^{-x^2}e^{-y^2}\,\mathrm{d}y = e^{-x^2}\int_{y=-\infty}^{y=+\infty} e^{-y^2}\,\mathrm{d}y\,.$$

(2) Thus $\mathrm{vol}(R) = \int_{x=-\infty}^{x=+\infty}\left[e^{-x^2}\left(\int_{y=-\infty}^{y=+\infty} e^{-y^2}\,\mathrm{d}y\right)\right]\mathrm{d}x = \left(\int_{y=-\infty}^{y=+\infty} e^{-y^2}\,\mathrm{d}y\right)\left(\int_{x=-\infty}^{x=+\infty} e^{-x^2}\,\mathrm{d}x\right) = \left(\int_{x=-\infty}^{x=+\infty} e^{-x^2}\,\mathrm{d}x\right)^2$.

3.1.2. Slicing in general. Let R be a "nice" subset of three-dimensional space. Fix an axis (say z), and suppose the points of R have z co-ordinates between a,b. Take a parition $P : a = z_0 < \cdots < z_n = b$ of $[a,b]$, and *slice*:

$$R = \bigcup_{i=1}^{n} R_i = \bigcup_{i=1}^{n}\left\{(x,y,z) \in R \mid z_{i-1} \le z \le z_i\right\}.$$

If $\delta(P)$ is small, the R_i have approximately constant cross-section. Write $R(z)$ for the *infinitesimal slice* $\{(x,y) \mid (x,y,z) \in R\}$ and $A(z)$ for its area. Let z_i^* be representative points in $[z_{i-1}, z_i]$. Then $\sum_{i=1}^{n} A(z_i^*)\Delta z_i$ is a Riemann sum. If $A(z)$ is Riemann integrable taking the limit will show that

$$\mathrm{vol}(R) = \int_a^b A(z)\,\mathrm{d}z\,.$$

Slicing paradigm:

(1) Choose an axis along which to slice. Slices will be *perpendicular* to the axis.
(2) Parametrize the solid, preferably so that the slicing axis is a co-ordinate axis, say the z-axis.
(3) Identify the cross-sections $R(z)$ and calculate their area $A(z)$
- Sometimes this is elementary.
- Sometimes we use geometry
- Sometimes we use integration to calculate the area.

(4) The volume is $\int_a^b A(z)\,\mathrm{d}z$.

EXAMPLE 121. Let C be a cone on base B and height H.

(1) Drop an altitude from the cone point to the plane containing the base. This altitute will be our z-axis.
(2) We may orient our axis so that the cone point is at $z = 0$ and that the base is at $z = H$.
(3) The slice at heigh z is then a rescaled copy of the base; by similarity it has area $\mathrm{Area}(B)\left(\frac{z}{H}\right)^2$.
(4) It follows that

$$\begin{aligned}\mathrm{vol}(C) &= \int_0^H \mathrm{Area}(B)\left(\frac{z}{H}\right)^2\mathrm{d}z\\ &= \frac{\mathrm{Area}(B)}{H^2}\int_0^H z^2\,\mathrm{d}z\\ &= \frac{1}{3}\mathrm{Area}(B)\cdot H\,.\end{aligned}$$

3.1.3. Solids of revolution (7/2/2012).

EXAMPLE 122. (Solid of revolution). Let R be the region obtained by revolving a planar set A around x-axis. Suppose that $A = \{(x,y) \mid x \in [a,b], 0 \le f(x) \le y \le g(x)\}$. Then

$$R = \left\{(x,y,z) \mid x \in [a,b], f(x) \le \sqrt{y^2+z^2} \le g(x)\right\}.$$

Slicing perpendicular to x-axis Each cross-section is an annulus of area $\pi\left(g^2 - f^2\right)$ so

$$\text{Area}(R) = \pi \int_a^b dx \left(g(x)^2 - f(x)^2\right).$$

For example, rotate the region $0 \le y \le \cos x$ about x-axis for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. The volume is

$$\pi \int_{-\pi/2}^{+\pi/2} \mathrm{d}x \cos^2 x = \frac{1}{2}\pi^2.$$

3.1.4. Other volume elements – cylindrical shells. Revolve $A = \{(x,y) \mid x \in [a,b], 0 \le f(x) \le y \le g(x)\}$ around y-axis. Renaming the axes the resulting region is $R = \Big\{(x,y,z) \mid a \le \sqrt{x^2+y^2} \le b, f(\sqrt{x^2+y^2}) \le z \le g($
Instead of slicing, divide R into cylndrical shells around the axis. The shell of radius r has surface area $2\pi r(g(r) - f(r))$ so the volume is

$$2\pi \int_a^b r\,\mathrm{d}r\,(g(r) - f(r)).$$

EXAMPLE 123.

(1) The volume of the ball is $\int_{x=0}^{x=R} \mathrm{d}x \left(2\sqrt{R^2-x^2} \cdot 2\pi x\right) = 2\pi \int_{u=0}^{u=R^2} \sqrt{R^2-u}\,\mathrm{d}u = -2\pi\left[-\frac{2}{3}(R^2-u)^{3/2}\right]_{u=0}^{u=R^2}$
$\frac{4\pi}{3}R^3$.

(2) A wineglass is obtained by revolving the area between the y-axis, the curve $y = x^2$ and the line $y = 4$ about the y-axis. Find the volume. $(2\pi \int_{x=0}^{2} x(4-x^2)\,\mathrm{d}x = 2\pi(2\cdot 4 - \frac{16}{4}) = 8\pi$. If the glass is half-full, how high is the liquid? We need to solve $2\pi \int_0^{\sqrt{H}} x(H-x^2)\,\mathrm{d}x = \frac{4\pi}{3}$ that is $2\pi\left(\frac{H^2}{2} - \frac{H^2}{4}\right) = 8\pi$. We get $H^2 = 8$, that is $H = 2\sqrt{2}$.

(3) $\left\{(x,y,z) \mid 0 \le z \le e^{-(x^2+y^2)}\right\}$ is a solid of revolution. Has volume $2\pi \int_{r=0}^{\infty} re^{-r^2} = \pi \int_{u=0}^{\infty} e^{-u}\,\mathrm{d}u = \pi$. Combining with Example 120 we find

$$\int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

3.2. Arc length (10/2/2012)

If f' is continuous, the length of the graph of $y = f(x)$ on $[a,b]$ is

$$\int_a^b \sqrt{1 + (f'(x))^2}\,\mathrm{d}x$$

(derived by cutting curve into approximate line segments).

3.3. Surface area (PS6)

3.4. Mass and density (13/2/2012)

DEFINITION 124. "Density" usually means amount of substance per geometric unit. Examples include:

- "Volume density of mass" = amount of mass per unit volume of the substance
- "Length density of charge" = amount of electrical charge per unit length of the material

"Density" with no other qualifiers will mean volume density of mass.

Suppose a body consists of material of variable density. Its total mass can then be calculated as a sum of contributions from different parts:

$$\text{Mass} = \int (\text{density}) \times \mathrm{d}\,\text{Vol}$$

where the volume elements are assumed to have constant density.

- Note: Most natural to slice at constant density.

EXAMPLE 125. A room has floor area A and height H. The density of air falls of like $\frac{\rho_0}{1+\frac{z}{l_0}}$ where z is the height above the floor (ρ_0 is the density of air at floor level, l_0 a length scale for the decay of the density). What is the mass of air in the room?

- Consider planar slices perpendicular to the z axis.
- The infinitesimal slice between heights $z, z+dz$ has infinitesimal volume $A\,\mathrm{d}z$ and hence infinitesimal mass $\frac{\rho_0}{1+\frac{z}{l_0}} A\,\mathrm{d}z$.
- The total mass is then

$$\int_0^H \frac{\rho_0 A}{1+\frac{z}{l_0}}\,\mathrm{d}z = (\rho_0 l_0 A)\int_0^H \frac{1}{1+\frac{z}{l_0}}\frac{\mathrm{d}z}{l_0} = (\rho_0 l_0 A)\log\left(1+\frac{H}{l_0}\right).$$

REMARK 126. Note that the *units* of work out correctly: $l_0 A$ has units of volume, ρ_0 has units of $\frac{\text{mass}}{\text{volume}}$ while $\frac{H}{l_0}$ has no units, so can be put in a log.

EXAMPLE 127. Perrin's law states that the density of air in the atmosphere decays like $\rho = \rho_0 e^{-\frac{r}{r_0}}$ where ρ_0 is the distance at the surface, r is the height above the surface, and r_0 is a length scale. What is the total mass of the atmosphere?

- Consider spherical shells around the Earth.
- The shell at height r and thickness dr has radius $(R+r)$ and therefore volume $4\pi(R+r)^2\,\mathrm{d}r$ (use formula for the area of the sphere). Hence the shell contributes the infinitesimal mass $4\pi\rho_0\,(R+r)^2\,e^{-\frac{r}{r_0}}\,\mathrm{d}r$.
- Integrating we find

$$\begin{aligned}
\text{Mass} &= 4\pi\rho_0 \int_0^\infty \left(R^2+2Rr+r^2\right)e^{-r/r_0}\,\mathrm{d}r \\
&= 4\pi\rho_0\left[R^2 r_0\int_0^\infty e^{-s}\,\mathrm{d}s + 2Rr_0^2\int_0^\infty s e^{-s}\,\mathrm{d}x + r_0^3\int_0^\infty s^2 e^{-s}\,\mathrm{d}s\right] \\
&= 4\pi\rho_0\left[R^2 r_0 + 2Rr_0^2 + 2r_0^3\right].
\end{aligned}$$

- Remark: since $r_0 \ll R$ (tens of km vs thousands of km) the first term dominates. In other words, most of the atmosphere is in the shell of thickness r_0 around the Earth, which has approximate volume $4\pi R^2 r_0$ and on which the density is about ρ_0. This is the "first order approximation" to which the other terms are corrections.

3.5. Centre-of-mass (14-15/2/2012)

3.5.1. 14/2/2012. Discussion: bodies resist forces according to mass, resist rotation according to moment of inertia. Most natural to rotate around the point where the moment is minimized, that is the center-of-mass.

DEFINITION 128. The x-co-ordinate of the *center of mass* is given by the weighted average:

$$\frac{\int x\rho \,\mathrm{d}\,\mathrm{Vol}}{\int \rho \,\mathrm{d}\,\mathrm{Vol}}$$

where $\mathrm{d}\,\mathrm{Vol}$ are volume elements and ρ is the density.

REMARK 129. In this course we will generally assume that ρ is a function of one co-ordinate only, and use symmetry to find the y,z co-ordinates of the center-of-mass.

EXAMPLE 130. A sword has length L, length density of mass $\frac{c}{z^2+l_0^2}$ where z is the distance from the hilt. Find the center of mass.

- It is at a distance from the hilt given by

$$\begin{aligned} \frac{\int_0^L z\frac{c}{z^2+l_0^2}\,\mathrm{d}z}{\int_0^L \frac{c}{z^2+l_0^2}\,\mathrm{d}z} &= \frac{\frac{1}{2}\int_{z=0}^{z=L}\frac{d(z^2+l_0^2)}{z^2+l_0^2}}{\frac{1}{l_0}\int_{z=0}^{z=L}\frac{d(z/l_0)}{1+(z/l_0)^2}} \\ &= \frac{l_0}{2}\cdot\frac{\log\left(\frac{L^2+l_0^2}{l_0^2}\right)}{\arctan\left(\frac{L}{l_0}\right)}. \end{aligned}$$

- But the sword is 3-dimensional?
  - Yes, but because of its reflection symmetry in x,y axes the center-of-mass must be along the axis of the sword, so can treat it as a 1-d problem. The "length density" above is really $\rho(z)A(z)$ where ρ is the volume density and $A(z)$ is the cross-sectional area of the sword.

EXAMPLE 131. The center of mass of some plane figures:

- The disc: by rotational symmetry (or by reflecting in x,y axes separate) the CM is in the center.
- An equilateral triangle: by symmetry again, this must be at the meeting point of the medians.
- An isoceles triangle:
  - *Symmetry*: The CM must be along the bisector of the angle at the meeting of the equal sides.
  - *Parametrize*: Assume that the base of the triangle is on the x-axis with the y-axis running along the bisector, so that the vertices at $(-a,0)$, $(a,0)$, $(0,H)$.

- *Slice:* Divide into strips perpendicular to the y-axis. By similarity of triangle the strip at height y from the base has length satsifying

$$\frac{L(y)}{H-y} = \frac{2a}{H}\,.$$

It follows that the center-of-mass is at

$$\frac{\int_0^H 2a\frac{H-y}{H}y\,\mathrm{d}y}{\int_0^H 2a\frac{H-y}{H}\,\mathrm{d}y}\,.$$

- *Integrate*: Note that a scales away, so that the length of the base doesn't matter. So can make triangle equilateral and calculate that way, or just do the integral:

$$= \frac{\left[H\frac{y^2}{2} - \frac{y^3}{3}\right]_0^H}{\left[Hy - \frac{y^2}{2}\right]_0^H} = \frac{\frac{1}{6}H^3}{\frac{1}{2}H^2} = \frac{1}{3}H\,.$$

- Remark: can simplify the calculation by having y axis run the opposite way, so that $y = 0$ at the tip.

3.5.2. 14/2/2012.

DEFINITION 132 (Euler's Gamma function). $\Gamma(s) = \int_0^\infty x^s e^{-x}\frac{\mathrm{d}x}{x}$.

Recall:

FACT 133 (PS5). *The integral converges absolutely for $s > 0$, satisfies the law $s\Gamma(s) = \Gamma(s+1)$, and hence $\Gamma(n+1) = n!$ for all natural numbers n.*

EXERCISE 134 (Change of variables). Show that $\Gamma(\frac{1}{2}) = \int_{-\infty}^{+\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}$.

EXAMPLE 135. Let R be the region under the graph of $y = e^{-x^2}$. Find its CM.

- By symmetry the CM must be on the y-axis, so we only need to find its y-co-ordinate.
- Hence we slice perpendicular to y-axis. Infinitesimal slices are horizontal strips extending from $(-x, y)$ to (x, y) where $y = e^{-x^2}$.
- It follows that the area of such a strip is $2\sqrt{-\log y}\,\mathrm{d}y$.
  - Sanity check: for us $0 < y \le 1$ so $\log y \le 0$ and $-\log y \ge 0$.
- The CM is therefore at height

$$\frac{2\int_0^1 \sqrt{-\log y}\,y\,\mathrm{d}y}{2\int_0^1 \sqrt{-\log y}\,\mathrm{d}y}\,.$$

We note the lower integral is simply the area of R, that is $\sqrt{\pi}$.

- We now evaluate the top integral. We get ride of the square root log by shifting to $y=e^{-t}$. We therefore need to calculate:

$$\begin{aligned}
\frac{2}{\sqrt{\pi}}\int_{\infty}^{0}\sqrt{t}e^{-t}(-e^{-t}\,\mathrm{d}t) &= \frac{1}{\sqrt{\pi}}\int_0^{\infty}\sqrt{t}e^{-(2t)}d(2t)\\
&= \frac{1}{\sqrt{2\pi}}\int_0^{\infty}x^{\frac{1}{2}}e^{-x}\,\mathrm{d}x\\
&= \frac{1}{\sqrt{2\pi}}\Gamma\left(\frac{3}{2}\right)=\frac{1}{\sqrt{2}\Gamma(\frac{1}{2})}\frac{1}{2}\Gamma(\frac{1}{2})\\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

PROBLEM 136. d

3.6. Integrating differential equations (17/2/2012)

Suppose $y(x)$ solves the equation $y'=G(x)F(y)$. Then by the change of variable formula, $\int\frac{\mathrm{d}y}{F(y)}=\int G(x)\,\mathrm{d}x$, and we can integrate and solve for y.

EXAMPLE 137. $y'=y$. Then $\frac{\mathrm{d}y}{y}=\mathrm{d}x$ so $\log|y|=x+C$. Thus $y=\pm e^{C}e^{x}$, which since y is continuous must take the form $y=Ae^{x}$, $A\in\mathbb{R}$, non-zero. Note that we don't find the solution $y=0$ since the change-of-variable makes no sense there, but setting $A=0$ in the formula does give this solution too.

EXAMPLE 138 (Logistic growth). Consider now the equation $y'=ry(1-y)$. Then $\frac{\mathrm{d}y}{y(1-y)}=r\,\mathrm{d}x$. Since $\frac{1}{y(1-y)}\sim_0\frac{1}{y}$ and $\frac{1}{y(1-y)}\sim_1\frac{1}{1-y}$ we can integrate using partial fractions:

$$\log\frac{y}{1-y}=rx+C,$$

at least in the interesting region where $0<y<1$ (boundaries correspond to total extinction and the maximal population, respectively. Then

$$\frac{y}{1-y}=e^{rx+C}$$

so

$$y=\frac{e^{rx+C}}{e^{rx+C}+1}=1-\frac{1}{e^{rx+C}+1},$$

where C is chosen so that $y(0)=1-\frac{1}{e^{C}+1}$.

EXAMPLE 139 (The Catenary). The equation $y''=C\sqrt{1+(y')^2}$ describes the shape of a hanging chain ("catenary" in the latin).

Letting $z=y'$ we have

$$\frac{dz}{\sqrt{1+z^2}}=Cdx.$$

We use the substitution ("hyperbolic sine") $z = \frac{e^t - e^{-t}}{2}$ (this is monotone for all t). Then $dz = \frac{e^t + e^{-t}}{2}$ and

$$\frac{1}{\sqrt{1+z^2}} = \frac{1}{\sqrt{1+\frac{e^{2t}-2+e^{-2t}}{4}}} = \frac{1}{\sqrt{\left(\frac{e^t+e^{-t}}{2}\right)^2}} = \frac{2}{e^t+e^{-t}}\,.$$

It follows that for some A we have

$$t = Cx + A$$

so that

$$z = \frac{e^{Cx+A} - e^{-(Cx+A)}}{2}\,.$$

We now integrate again to find:

$$y = \frac{1}{2C}e^{Cx+A} + \frac{1}{2C}e^{-(Cx+A)} + B\,.$$

Suppose now that the chain hangs with endpoints at $x = \pm L$, $y = 0$. Then by symmetry we must have $A = 0$ and $B = -\frac{e^{CL}+e^{-CL}}{2C}$. It follows that

$$y = \frac{e^{Cx}+e^{-Cx}}{2C} - \frac{e^{CL}+e^{-CL}}{2C}\,.$$

The bottom of the chain, for example, is at height $-\frac{(e^{CL/2}-e^{-CL/2})^2}{2}$

3.7. Continuous Probability (27-29/2)

3.7.1. Discrete Probability.

3.7.1.1. *Problems to keep in mind:*

- We roll a die. How much would you bet that the die comes out $1,2$ or 3?
- A gambling game is interrupted in the middle. How do we divide the pot?

3.7.1.2. *Probability spaces.*

NOTATION 140. Ω will be a set ("sample space" or "space of simple events"). An *event* will mean a subset of Ω.

DEFINITION 141. A real-valued function Pr defined on subsets of Ω is called a *probability distribution* if:

(1) For $A \subset \Omega$, $0 \le \Pr(A) \le 1$
(2) $\Pr(\emptyset) = 0$ and $\Pr(\Omega) = 1$.
(3) If A, B are disjoint events then $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

REMARK 142. For technical reasons not all subsets may have a probability and we need (3) to apply to infinite disjoint unions too.

- We suppose for now that Ω is finite.

EXAMPLE 143 (Uniform distribution). Let Ω be finite, and for $A \subset \Omega$ set $\Pr(A) = \frac{\#A}{\#\Omega}$. Easy to check (1),(2),(3) above (the third is the statement that the size of a disjoint union is a sum of the sizes of the parts.

If $\Omega = \{1, \cdots, 6\}$ and $A = \{1,2\}$, $B = \{3\}$ then $\Pr(A) = \frac{1}{3}$, $\Pr(B) = \frac{1}{6}$ and $\Pr(A \cup B) = \frac{1}{2}$.

More generally, a probability distribution Pr on a finite set Ω is determined by the values at simple events, $p_\omega = \Pr(\{\omega\})$. Indeed each p_ω is a number between $0,1$ subject to $\sum_{\omega\in\Omega} p_\omega = \Pr\left(\cup_{\omega\in\Omega}\{\omega\}\right) = \Pr(\Omega) = 1$ and for any A, $\Pr(A) = \sum_{\omega\in A} p_\omega$.

EXAMPLE 144. A simple lottery has $\Omega = \{\text{win, loss}\}$ with $p_{\text{win}} = 10^{-6}$ and $p_{\text{loss}} = 1-10^{-6}$.

3.7.1.3. *Random variables, expectation, variance etc.*

DEFINITION 145. A *random variable* is a real-valued function on X.

EXAMPLE 146. For a final example, let Ω be the set of UBC students, with the uniform probability distribution. Let $X(\omega)$ be the height of the student, $Y(\omega)$ the GPA of the student.

DEFINITION 147. The *expectation* of X is defined to be the weighted average $\mathbb{E}X \stackrel{\text{def}}{=} \sum_{\omega\in\Omega} p_\omega X(\omega)$. When X is clear we write $\mu = \mathbb{E}X$.

EXAMPLE 148. Consider the uniform distribution on $\Omega = \{1,\cdots,6\}$ and the two random variables $X(\omega) = \omega$ and $Y(\omega) = 2^\omega$. We have

$$\mathbb{E}X = \sum_{i=1}^{6} \frac{1}{6} i = 3.5$$

and

$$\mathbb{E}Y = \sum_{i=1}^{6} \frac{1}{6} 2^i = \frac{1}{6} 2 \sum_{i=0}^{5} 2^i = \frac{1}{3}\left(2^6 - 1\right) = 21\,.$$

DEFINITION 149. The *Variance* of X is the expectation of the random variable $(X-\mu)^2$. We calculate:

$$\operatorname{Var}(X) = \mathbb{E}\,(X-\mu)^2 = \mathbb{E}\left(X^2 - 2\mu X + \mu^2\right) = \mathbb{E}(X^2) - 2\mu\mathbb{E}X + \mu^2 = \mathbb{E}(X^2) - (\mathbb{E}X)^2\,.$$

The *standard deviation* of X is $\sigma = \sqrt{\operatorname{Var}(X)}$.

3.7.2. Continuous probability.

3.7.2.1. *Definition.*

DEFINITION 150. We say that the random variable X has a *continuous distribution* if there is a function $p(x)$ so that

$$\Pr(a \le X \le b) = \int_a^b p(x)\,\mathrm{d}x\,.$$

In that case p is called the *probability density* function.

LEMMA 151. *p is a probability density function if and only if:*

(1) *$p(x) \ge 0$ for all x.*
(2) *$\int_{-\infty}^{+\infty} p(x)\,\mathrm{d}x = 1$.*

- Normalization: suppose f is non-negative and not identically zero, and $C = \int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x < \infty$. Then $p(x) = \frac{f(x)}{C}$ is a probability density function.

EXAMPLE 152 (Uniform continuous distribution). Let $f(x) = \begin{cases} 1 & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$. This expresses the idea that x is uniformly distributed between a, b. We first normalize: $\int_{\mathbb{R}} f(x)\,\mathrm{d}x = b - a$ so

$$p(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

is the probability distribution function.

PROBLEM 153. X is chosen uniformly at random in $[5, 12]$. What is the probability that $3 \le X \le 7$?

Solution: this is $\int_3^7 \frac{1}{7}\,\mathrm{d}x = \frac{4}{7}$.

EXAMPLE 154 (Exponential distribution). Suppose that X has a distribution with density function proportional to e^{-ax} for $x \ge 0$, zero otherwise. What is the probability density function?

Solution: $\int_0^\infty e^{-ax}\,\mathrm{d}x = \frac{1}{a}$ so the PDF is $p(x) = ae^{-ax}$.

3.7.2.2. *Expectation, Variance etc.*

DEFINITION 155. Let $p(x)$ be the probability density function of a random variable X.

(1) If $\int_{\mathbb{R}} xp(x)\,\mathrm{d}x$ converges, we say that X has an expectation and set $\mu = \mathbb{E}X = \int_{-\infty}^{+\infty} xp(x)\,\mathrm{d}x$.
(2) If $\int_{-\infty}^{+\infty} x^2 p(x)\,\mathrm{d}x$ we say that X has variance, and also set $\sigma^2 = \mathbb{E}(X-\mu)^2 = \int_{-\infty}^{+\infty}(x-\mu)^2 p(x)\,\mathrm{d}x = \int_{-\infty}^{+\infty} x^2 p(x)\,\mathrm{d}x - \mu^2$.

EXAMPLE 156. For the uniform distribution on $[a, b]$, the expectation is $\int_a^b x \frac{\mathrm{d}x}{b-a} = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$. The variance is

$$\int_a^b x^2 \frac{\mathrm{d}x}{b-a} - \left(\frac{b+a}{2}\right)^2 = \frac{b^3-a^3}{3(b-a)} - \frac{a^2+2ab+b^2}{4} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}.$$

It follows that $\sigma = \frac{b-a}{\sqrt{12}}$.

3.7.2.3. *Example: the exponential distribution.*

CHAPTER 4

Parametric curves

See textbook.
Example: $t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$ traces the unit circle except for the point $(0,1)$.

CHAPTER 5

Sequences and Series

5.1. Sequences and Convergence

5.1.1. Review: sequences (6/3/2012).

DEFINITION 157. A *sequence* is a function whose domain is $\{n \in \mathbb{Z} \mid n \geq n_0\}$.

EXAMPLE 158. The following are sequences of functions

- $f_n(x) = x^n$ on $[0,1]$, for $n = 0,1,2,\ldots$
- $g_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$.

DEFINITION 159. (Arithmetic of sequences) Let $\{a_n\}_{n\geq n_0}$, $\{b_n\}_{n\geq n_0}$ be sequences defined on the same set of integers. We call the sequences $\{a_n + b_n\}_{n\geq n_0}$, $\{a_n - b_n\}_{n\geq n_0}$, $\{a_n \cdot b_n\}_{n\geq n_0}$ respectively the *sum*, *difference*, and *product* of the two sequences.

DEFINITION 160. Let $\{a_n\}_{n\geq n_0} \subset \mathbb{R}$ be a sequence.

- We say that it is *increasing* if $a_{n+1} > a_n$ for all $n \geq n_0$, *non-decreasing* if $a_{n+1} \geq a_n$. *Decreasing* and *non-increasing* sequences are defined similarly. In any of the cases we say the sequence is *monotone*.
- We say the sequence is *bounded above* if for some M, $a_n \leq M$ for all M. We say the sequence is *bounded below* if for some m we have $a_n \geq m$ for all m. We say the sequence is *bounded* if it is bounded above and below, equivalently if $\{|a_n|\}_{n\geq n_0}$ is bounded above.

DEFINITION 161. A *tail* of the sequence $\{a_n\}_{n\geq n_0}$ is any sequence of the form $\{a_n\}_{n\geq n_1}$ where $n_1 \geq n_0$.

EXAMPLE 162. $5,6,7,8,9,\cdots$ is a tail of $0,1,2,3,4,5,\cdots$.

DEFINITION 163. We say the sequence $\{a_n\}_{n\geq n_0}$ *eventually* has some property if some tail of the sequence has it.

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EXAMPLE 164. Some explicit sequences:

(1) $a_n = 1$ for $n \geq 0$.
(2) $b_n = 1 + \frac{1}{n}$ for $n \geq 1$
(3) $c_n = \begin{cases} 1 & n \text{ prime} \\ 0 & \text{otherwise} \end{cases}$
(4) $d_n = e^{-n}$ for $n \geq -5$.
(5) $e_n = n$ for $n \geq 0$
(6) $f_n = n$ for $n \geq 1$.
(7) $g_n = 1 - \frac{1}{n}$.

EXERCISE 165. Which of the sequences are increasing? non-decreasing? decreasing? non-increasing?

EXERCISE 166. Which of the sequences is bounded above? bounded below? bounded? Find explicit bounds when applicable.

EXERCISE 167. Show that $h_n = (n-5)^2$ for $n \geq 0$ is eventually increasing. Show that $T_n = \frac{1}{1+(n-5)^2}$ is eventually decreasing. Show that $R_n = n^3 - 10$ is eventually positive. Show that b_n is eventually less than $1 + \frac{1}{10^6}$.

5.1.2. Limits (7/3/2012).

DEFINITION 168. Let $\{a_n\}_{n\geq n_0} \subset \mathbb{R}$ be a sequence of real numbers. Let $L \in \mathbb{R}$. We say that "the sequence tends to the limit L" and write $\lim_{n\to\infty} a_n = L$ if for every $\varepsilon > 0$, eventually the a_n are in $[L-\varepsilon, L+\varepsilon]$. Equivalently, if for every $\varepsilon > 0$ there is N such that for $n \geq N$ we have $|a_n - L| \leq \varepsilon$. If the sequence tends to a limit we say that it *converges*. It it does not tend to any limit we say that it *diverges*.

EXAMPLE 169. Let $a_n = A$ for all n. Then $\lim_{n\to\infty} a_n = A$.

PROOF. Given $\varepsilon > 0$ let $N = 0$. Then if $n \geq 0$ we have $|a_n - A| = 0 \leq \varepsilon$. $\square$

EXAMPLE 170. Let $b_n = \frac{1}{n}$ for all $n \geq 1$. Then $\lim_{n\to\infty} a_n = 0$.

PROOF. Indeed for $\varepsilon > 0$ let $N = \frac{1}{\varepsilon}$. then if $n \geq N$ we have $|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} = \varepsilon$. $\square$

EXAMPLE 171. The sequence $c_n = (-1)^n$ diverges.

PROOF. Suppose $\lim_{n\to\infty} c_n = L$ was true. Then for $\varepsilon = \frac{1}{2}$ there would be N so that for $n \geq N$, $|c_n - L| \leq \frac{1}{2}$. Taking an even $n \geq N$ we see that $|1-L| \leq \frac{1}{2}$. Taking and odd $n \geq N$ we also see that $|-1-L| \leq \frac{1}{2}$. It follows that $|2| \leq |1+L| + |1-L| \leq 1$, a contradiction. $\square$

LEMMA 172. *A sequence can have at most one limit.*

PROOF. Suppose $\lim_{n\to\infty} a_n = A$ and also $\lim_{n\to\infty} a_n = B$ and that $A < B$. Let $\varepsilon < \frac{B-A}{2}$. Then there are N_1, N_2 so that for $n \geq N_1$, $a_n \in [A-\varepsilon, A+\varepsilon]$ and for $n \geq N_2$, $a_n \in [B-\varepsilon, B+\varepsilon]$. Now for $n \geq \max\{N_1, N_2\}$ we see that $a_n \leq A + \varepsilon < B - \varepsilon \leq a_n$, a contradiction. $\square$

LEMMA 173. *Let $\{a_n\}_{n\geq n_1}$ be a tail of $\{a_n\}_{n\geq n_0}$. Then either both converge or both diverge; in the first case they have the same limit.*

PROPOSITION 174. *Let $\{a_n\}_{n\geq n_0}$converge. Then it is bounded.*

PROOF. Suppose $\lim_{n\to\infty} a_n = L$, and let N be such that for $n \geq N$ we have $|a_n - L| \leq 1$. Let $M = \max\{|L|+1, |a_{n_0}|, |a_{n_0+1}|, \cdots, |a_N|\}$. The for any n we have $|a_n| \leq L$ (either $n \leq N$ or $n \geq N$). $\square$

COROLLARY 175. *$\lim_{n\to\infty} n$ does not exist.*

ALGORITHM 176 (Arithmetic of limits). *Suppse $\{a_n\}_{n\geq n_0}, \{b_n\}_{n\geq n_0} \subset \mathbb{R}$ are convergent sequences with $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$.*

(1) *(linearity) Let $\alpha, \beta \in \mathbb{R}$. Then $\lim_{n\to\infty}(\alpha a_n + \beta b_n)$ exists and equals $\alpha A + \beta B$*
(2) *(multiplicativity) $\lim_{n\to\infty}(a_n b_n)$ exists and equals AB.*
(3) *Suppose $B \neq 0$. Then b_n is eventually non-zero, and $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and equals $\frac{A}{B}$.*

PROOF. We have $|(\alpha a_n + \beta b_n) - (\alpha A + \beta B)| \leq |\alpha| |a_n - A| + |\beta| |b_n - B|$ and $|a_n b_n - AB| \leq |a_n - A| |b_n| + |A| |b_n - B| \leq |a_n - A| M + |A| |b_n - B|$ where M is a bound for $|b_n|$. Finally, eventually $|b_n - B| \leq \frac{1}{2}|B|$ so $|b_n| \geq \frac{1}{2}|B| > 0$ and after that point $\left|\frac{1}{b_n} - \frac{1}{B}\right| = \frac{|B-b_n|}{|b_n B|} \leq \frac{2}{|B|^2}|b_n - B|$. $\square$

EXAMPLE 177. $\lim_{n\to\infty} \frac{n^7+8n+1}{3n^7-2n^2} = \lim_{n\to\infty} \frac{1+8n^{-6}+n^{-7}}{3-2n^{-5}} = \frac{1}{3}$.

To know that some limits exists we need a rule that creates real numbers.

AXIOM 178. *Let A, B be non-empty sets of real numbers such that if $a \in A$, $b \in B$ then $a \le b$. Then there is a real number L such that $a \le L \le b$ for all $a \in A$, $b \in B$.*

THEOREM 179. *Let $\{a_n\}_{n=1}^{\infty} \subset \mathbb{R}$ be monotone and bouned. The sequence then converges.*

PROOF. Suppose that a_n is non-decreasing. Let A be the set of values taken by the sequence, and let B be the set of upper bounds for the sequence. Then by assumption A, B are non-empty and satisfy the hypotheses of the axiom. Let L be as in the axiom. Then $a_n \le L$ for all L. Now let $\varepsilon > 0$. Then $L - \varepsilon < L$ so $L - \varepsilon \notin B$. It follows that there is N so that $a_N > L - \varepsilon$. Then for $n \ge N$ we have $L - \varepsilon < a_N \le a_n \le L$. It follows that $|a_n - L| \le \varepsilon$ if $n \ge N$. $\square$

EXAMPLE 180. Suppose that $a_{n+1} = \sqrt{a_n + 2}$. Find

SOLUTION. First, suppose that $L = \lim_{n\to\infty} a_n$ exists. Then $\lim_{n\to\infty}(a_n + 2) = L + 2$ by arithmetic of limits, and $\lim_{n\to\infty} \sqrt{a_n + 2} = \sqrt{L+2}$ by continuity of $\sqrt{\cdot}$. Since $\{a_{n+1}\}$ is a tail of a_n it also tends to the same limit and we find $L = \sqrt{L+2}$ so $L^2 - L - 2 = 0$ with solutions $L = 2, -1$. Since $a_n \ge 0$ for all n we must have $L = 2$. We now show that the sequence is bounded and monotone. First, suppose $a_n < 1000$. Then $a_{n+1} \le \sqrt{1000+2} < 1000$ so by induction $0 \le a_n \le 1000$ for all n. Next,

$$a_{n+1} - a_n = \sqrt{a_n + 2} - \sqrt{a_{n-1} + 2} = \frac{(a_n + 2) - (a_{n-1} + 2)}{\sqrt{a_n + 2} + \sqrt{a_{n-1} + 2}} = \frac{a_n - a_{n-1}}{\sqrt{a_n + 2} + \sqrt{a_{n-1} + 2}}.$$

It follows that $a_{n+1} - a_n$ has constant sign, so the sequence is monotone. It is therefore convergent, to 2.

Alternative: $2 - a_{n+1} = \frac{2 - a_n}{2 + \sqrt{a_n + 2}} < \frac{2 - a_n}{2}$ so $0 < 2 - a_{n+1} < \frac{2 - a_1}{2^n}$. It follows that $a_n \to 2$. $\square$

EXAMPLE 181. Let $a_0 = 1$ and let $a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n}$. Then by induction:

(1) Suppose it converged. Then the limit would satisfy $L = \frac{L}{2} + \frac{1}{L}$, so $L = \sqrt{2}$.
(2) $a_{n+1} - a_n = \frac{a_n - a_{n-1}}{2} + \frac{1}{a_n} - \frac{1}{a_{n-1}} = (a_n - a_{n-1})\left[\frac{1}{2} - \frac{1}{a_n a_{n-1}}\right] = (a_n - a_{n-1})\left[\frac{1}{2} - \frac{1}{1 + a_{n-1}^2/2}\right]$
(3) $a_n > 0$ for all n.
(4) $\sqrt{2} - a_{n+1} = \sqrt{2} - \frac{a_n}{2} - \frac{1}{a_n} = \frac{\sqrt{2} - a_n}{2} + \frac{1}{\sqrt{2}} - \frac{1}{a_n} = \left(\sqrt{2} - a_n\right)\left[\frac{1}{2} - \frac{1}{\sqrt{2}a_n}\right]$. It follows that $a_n < \sqrt{2}$ for all n.
(5) It follows that $\frac{1}{2} + \frac{1}{\sqrt{2}a_n} < 1$ so $\sqrt{2} - a_{n+1} \le \sqrt{2} - a_n$. The sequence is therefore monotone and bounded.

THEOREM 182. *(Squeeze) Suppose that eventually $a_n \le b_n \le c_n$ and that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$. Then $\lim_{n\to\infty} b_n = L$.*

PROOF. Given $\varepsilon > 0$ we know that eventually $a_n \ge L - \varepsilon$ and $c_n \le L + \varepsilon$. It follows that eventually $L - \varepsilon \le b_n \le L + \varepsilon$. $\square$

Math 121: In-class worksheet for lecture

EXAMPLE 183. Decide whether each sequences converges. If so, determine the limit.

(1) $a_n = \frac{1}{n^2+1}$, $n \geq 0$.
(2) $b_n = \frac{n}{n^2+1}$, $n \geq 0$
(3) $c_n = \frac{n^2}{n^2+1}$, $n \geq 0$
(4) $d_n = \frac{n^3 \cos n}{n^2+1}$, $n \geq 0$.

EXERCISE 184. Let $x, y \geq 0$. Show that $\frac{x+y}{2} \geq \sqrt{xy}$.

EXERCISE 185. Let $0 \le a_0 \le b_0$ be given. Define $a_{n+1} = \sqrt{a_n b_n}$ and $b_{n+1} = \frac{a_n+b_n}{2}$.

(1) Show that $0 \le a_n \le b_n$ for all n.
(2) Show that $\{a_n\}_{n=0}^{\infty}$ is non-decreasing and that $\{b_n\}_{n=0}^{\infty}$ is non-increasing.
(3) Show that the sequences are bounded.
(4) (**)Show that $b_{n+1} - a_{n+1} \le \frac{b_n - a_n}{2}$
(5) Show that $0 \le b_n - a_n \le \frac{b_0 - a_0}{2^n}$.
(6) Conclude that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.
Hint: Apply the Squeeze thm to the conclusion of (5).

5.2. Series

For exam: you need not prove that $\frac{1}{n}$, $\frac{1}{n^2}$, q^n for $|q| < 1$ all converge to zero, unless you are directly asked to prove that.

5.2.1. Definitions, exactly summable cases.

DEFINITION 186. A *series* is a formal sum $\sum_{n=n_0}^{\infty} a_n$ where $\{a_n\}_{n=n_0}^{\infty} \subset \mathbb{R}$ is a sequence. A *partial sum* of the series is a sum $s_n = \sum_{k=n_0}^{k=n} a_k$. We say that the series *converges* if $S = \lim_{n\to\infty} s_n$ exists, in which case we call the limit the *sum* of the series and write $\sum_{n=n_0}^{\infty} a_n = S$. If the series does not converge we say it *diverges*.

EXAMPLE 187. Exactly summable series

- $\sum_{n=0}^{\infty} 0$, $\sum_{n=0}^{\infty} 1$, $\sum_{n=0}^{\infty} (-1)^n$.
- For $q \neq 1$ consider $\sum_{n=0}^{\infty} q^n$. The partial sums (see PS1) are $\frac{q^n-1}{q-1}$. If $|q| > 1$ we get divergence, if $q < 1$ we converge to $\frac{1}{1-q}$.
- Consider $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Partial fractions: $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ so $s_n = 1 - \frac{1}{n+1}$. In particular the series sum to 1.

Tail estimates. Suppose $S = \sum_{n=0}^{\infty} a_n$. We the use partial sums to *approximate* S to better and better degree. Specifically, $S = a_0 + a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} a_n$, so bounding $\left|\sum_{n=N+1}^{\infty} a_n\right|$ esimates the *error* in $S \approx a_0 + a_1 + \cdots + a_N$. This is called making a *tail estimate*.

EXAMPLE 188. $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$. Thus $\log\left(\frac{2}{3}\right) = -\sum_{n=1}^{\infty} \frac{1}{n} 3^{-n}$. If we want to approximate $\log\left(\frac{2}{3}\right)$ to within 10^{-8} we need to find N so that $\left|\sum_{n=N+1}^{\infty} \frac{1}{n} 3^{-n}\right| \leq 10^{-8}$. Note that $\sum_{n=N+1}^{\infty} \frac{1}{n} 3^{-n} \leq \frac{1}{(N+1)3^{N+1}} \sum_{k=0}^{\infty} 3^{-k}$ so an upper bound on the error is $\frac{1}{(N+1)3^{N+1}} \frac{1}{1-\frac{1}{3}} = \frac{1}{2(N+1)3^N}$. Now we have $3^7 \geq 1000$ so $3^{14} \geq 10^6$ and $3^{20} \geq 10^8$. Smaller values of N are enough. Note that we need about $-\log \varepsilon$ terms for accuracy ε.

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EXAMPLE 189. $\frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} \cdots$. Now to get accuracy ε need about $\frac{1}{\varepsilon}$ terms. Convergence is too slow for the series to be useful.

THEOREM 190. *(Arithemtic of series) Suppose $\sum_n a_n$, $\sum_n b_n$ converge, and let $\alpha, \beta \in \mathbb{R}$. Then $\sum_n (\alpha a_n + \beta b_n)$ converges and its sum is $\alpha \sum_n a_n + \beta \sum_n b_n$.*

PROOF. $\sum_{n=1}^{T} (\alpha a_n + \beta b_n) = \alpha \sum_{n=1}^{T} a_n + \beta \sum_{n=1}^{T} b_n$. Now let $T \to \infty$ and use arithmetic of limits. □

5.2.2. Positive series and comparison.

DEFINITION 191. Call $\sum_n a_n$ *positive* if $a_n \geq 0$ for all n.

PROPOSITION 192. *A positive series converges iff its partial sums are bounded.*

PROOF. The partial sums form a monotone sequence. □

THEOREM 193 (Comparison test). *Let $0 \leq a_n \leq b_n$. If $\sum_n b_n < \infty$ then $\sum_n a_n < \infty$ and $\sum_n a_n \leq \sum_n b_n$. Conversely, if $\sum_n a_n = \infty$ then $\sum_n b_n = \infty$.*

PROOF. Bound the partial sums. □

THEOREM 194 (Integral test). *Let $f(x)$ be continuous. Suppose that f is non-increasing and non-negative for $x \geq N$. Then $\sum_{n=N}^{\infty} f(n)$ converges if and only if $\int_N^{\infty} f(x)\,\mathrm{d}x$.*

PROOF. Consider the following three regions in the plane: $R_1 = \{(x,y) \mid x \geq N, 0 \leq y \leq f(\lfloor x \rfloor)\}$, $R_2 = \{(x,y) \mid x \geq N, 0 \leq y \leq f(x)\}$, $R_3 = \{(x,y) \mid x \geq N, 0 \leq y \leq f(\lceil x \rceil)\}$. Since $\lfloor x \rfloor \leq x \leq \lceil x \rceil$ for all x we have $f(\lfloor x \rfloor) \geq f(x) \geq f(\lceil x \rceil)$ so $R_1 \supset R_2 \supset R_3$. Write $R_i(T) = \{(x,y) \in R_i \mid x \leq T\}$ for the truncation at T. If $T > N$ is an integer we then have:

$$\mathrm{Area}(R_1(T)) \geq \mathrm{Area}(R_2(T)) \geq \mathrm{Area}(R_3(T))$$

that is

$$\sum_{n=N}^{T-1} f(n) \geq \int_N^T f(x)\,\mathrm{d}x \geq \sum_{n=N+1}^{T} f(n).$$

Now if the integral converges then $\sum_{n=N+1}^{T} f(n) \leq \int_N^T f(x)\,\mathrm{d}x \leq \int_N^{\infty} f(x)\,\mathrm{d}x$ so the partial sums of the series are all bounded and the series converges. Conversely, if the series converges then $\int_N^T f(x)\,\mathrm{d}x \leq \sum_N^{\infty} f(n)$ for all T so by the monotone convergence principle for functions $\int_N^{\infty} f(x)\,\mathrm{d}x$ exists. □

EXAMPLE 195. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$ since this is the case for $\int_1^{\infty} \frac{\mathrm{d}x}{x^p}$.

DEFINITION 196. The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ is called the *harmonic series*. Its divergence is important.

5.3. Absolute convergence

DEFINITION 197. Say $\sum_n a_n$ *converges absolutely* if $\sum_n |a_n| < \infty$.

PROPOSITION 198. *Absolutely convergent series converge.*

PROOF. For every n we have $-|a_n| \leq a_n \leq |a_n|$ so $0 \leq a_n + |a_n| \leq 2|a_n|$. If $\sum_{n=n_0}^{\infty} |a_n|$ converges then by Theorem 190 so does $\sum_{n=n_0}^{\infty} (2|a_n|)$. By Theorem 193, $\sum_{n=n_0}^{\infty} (a_n + |a_n|)$ converges as well. Theorem 190 now gives the convergence of $\sum_{n=n_0}^{\infty} ((a_n + |a_n|) - |a_n|)$. □

THEOREM 199 (d'Alambert's criterion). *Suppose for all n large enough we have $|a_{n+1}| \leq \eta |a_n|$ where $0 \leq \eta < 1$. Then $\sum_{n=0}^{\infty} a_n$ converges absolutely.*

PROOF. Suppose that the property holds for $n \geq n_0$. Then $\left|a_{n_0+k}\right| \leq \eta^k |a_{n_0}|$ so $\sum_{n=n_0}^{\infty} |a_n|$ converges by comparison to the geometric series $\sum_{n=n_0}^{\infty} |a_{n_0}| \eta^{n-n_0}$. □

EXAMPLE 200. $\sum n2^{-n}$, $\sum \frac{1}{n2^n}$, $\sum n!x^n$, $\sum \frac{x^n}{n!}$.

DEFINITION 201. $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

THEOREM 202. *Let $\sum_{n \geq n_0} a_n$ converge absolutely. Then any rearrangement of it converges, and to the same sum.*

THEOREM 203. *Let $\sum_{m \geq m_0} a_m$, $\sum_{n \geq n_0} b_n$ converge absolutely with sums A, B respectively. Then $\sum_{m \geq m_0, n \geq n_0} a_n b_m$ converges absolutely and its sum is AB.*

5.3.1. Playing with power series: product of series.

LEMMA 204 (Binomial formula). $(x+y)^n = \sum_{k+l=n} \frac{n!}{k!l!} x^k y^l$.

PROOF. Induction. $\square$

THEOREM 205. $(\cos x)^2 + (\sin x)^2 = 1$.

PROOF. We have

$$\begin{aligned}
(\cos x)^2 &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}\right)\left(\sum_{l=0}^{\infty} \frac{(-1)^l x^{2l}}{(2l)!}\right) \\
&= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} x^{2(k+l)}}{(2k)!(2l)!} \\
&= \sum_{n=0}^{\infty} \sum_{k+l=n} \frac{(-1)^{k+l} x^{2(k+l)}}{(2k)!(2l)!} \\
&= \sum_{n=0}^{\infty} (-1)^n x^{2n} \sum_{k+l=n} \frac{1}{(2k)!(2l)!}\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\sin x)^2 &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}\right)\left(\sum_{l=0}^{\infty} \frac{(-1)^l x^{2l+1}}{(2l+1)!}\right) \\
&= \sum_{k,l=0}^{\infty} \frac{(-1)^{k+l} x^{2(k+l+1)}}{(2k+1)!(2l+1)!} \\
&= \sum_{n=1}^{\infty} (-1)^{n-1} x^{2n} \sum_{k+l=n-1} \frac{1}{(2k+1)!(2l+1)!}\,.
\end{aligned}$$

Note that $k+l=n-1$ is equivalent to $(2k+1)+(2l+1)=2n$, so we can summarize the calculation as:

$$\begin{aligned}
(\cos x)^2 &= 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} \sum_{\substack{a+b=2n \\ a,b \text{ even}}} \frac{1}{a!b!} \\
(\sin x)^2 &= - \sum_{n=1}^{\infty} (-1)^n x^{2n} \sum_{\substack{a+b=2n \\ a,b \text{ odd}}} \frac{1}{a!b!}
\end{aligned}$$

so

$$\begin{aligned}
(\cos x)^2 + (\sin x)^2 &= 1 + \sum_{n=1}^{\infty} (-1)^n x^{2n} \sum_{a+b=2n} \frac{1}{a!b!}(-1)^b \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} x^{2n} \sum_{a+b=2n} \binom{2n}{a} 1^a (-1)^b \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n!} x^{2n} \left(1+(-1)\right)^{2n} \\
&= 1\,.
\end{aligned}$$

$\square$

EXERCISE 206. $\cos(x+y) = \cos x \cos y - \sin x \sin y$. Also, $\sin(x+y) = \sin x \cos y + \cos x \sin y$.

LEMMA 207 (More tail estimates). $\lim_{x\to 0} \frac{\sin x}{x} = 1$, $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = -\frac{1}{2}$.

PROOF. $\frac{\sin x}{x} - 1 = \sum_{k=1}^{\infty}(-1)^k \frac{x^{2k}}{(2k+1)!}$ so if $|x| < 1$, $\left|\frac{\sin x}{x} - 1\right| \leq \frac{1}{3!}\sum_{k=1}^{\infty}|x|^{2k} = \frac{1}{6}\cdot\frac{x^2}{1-x^2}$. Now $\lim_{x\to 0}\frac{x^2}{1-x^2} = 0$ and the first claim follows from the squeeze theorem. The second argument is similar. $\square$

PROPOSITION 208. $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$, $\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$.

PROOF. $\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0}\left[\frac{\cos h - 1}{h}\sin x + \frac{\sin h}{h}\cos x\right] = \sin x\left(\lim_{h\to 0}\frac{\cosh - 1}{h^2}\right)\left(\lim_{h\to 0} h\right) + \cos x\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = \cos x$. The calculation for cosine is similar. $\square$

COROLLARY 209. *The functions* $\cos x, \sin x$ *are everywhere infinitely differentiable, hence continuous.*

DEFINITION 210. $\frac{\pi}{2} \stackrel{\text{def}}{=} \min\{x > 0 \mid \cos x = 0\}$.

The set is non-empty since $\cos 0 = 1$ while $\cos 2 < 0$ (homework).

THEOREM 211. $\cos(x+2\pi) = \cos x$, $\sin(x+2\pi) = \sin x$ *and there is no smaller period.*

PROOF. The first claim follows from the addition rule and the half-angle formula. For the second claim note that a period must divide 2π. But $\cos x$ is non-zero in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (it is even, and has no zeroes before $\frac{\pi}{2}$ by definition). It follows that the shortest possible period is π, which isn't the case since $\cos(x+\pi) = -\cos x$. $\square$

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5.4. Power series and Taylor series

5.4.1. Power series.

DEFINITION 212. A *power series* is a series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. The a_n are called the *coefficients* and x_0 is called the *centre*. The set of x where the series converges is called the *interval of convergence*.

PROPOSITION 213. *Suppose that the power series converges at some* $x_1 \neq x_0$. *Then it converges absolutely in* $\{x \mid |x-x_0| < |x_1-x_0|\}$.

PROOF. Since $\sum_{n=0}^{\infty} a_n(x_1-x_0)^n$ converges, we have $\lim_{n\to\infty} a_n(x_1-x_0)^n = 0$. In particular, this sequence is bounded. Suppose $|a_n(x_1-x_0)^n| \leq A$ for all n. Then $|a_n(x-x_0)^n| \leq A\left|\frac{x-x_0}{x_1-x_0}\right|^n$. If $|x-x_0| < |x_1-x_0|$ then $\sum_{n=0}^{\infty}\left|\frac{x-x_0}{x_1-x_0}\right|^n$ converges, so the original series converges absolutely. $\square$

COROLLARY 214. $\{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-x_0)^n$ *converges*$\}$ *has one of the following forms:*

(1) $\{x_0\}$
(2) *For some* $R > 0$, $(x_0-R, x_0+R) \cup E$ *where* $E \subset \{x_0-R, x_0+R\}$

(3) $\mathbb{R}$

DEFINITION 215. R is called the *radius of convergence*.

EXAMPLE 216. Suppose $\sum_{n=0}^{\infty} a_n(x-3)^n$ converges at $x=-5$. Then it must converge absolutely at $x=7$, but nothing is known about $x=15$.

THEOREM 217. *Suppose $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ exists in the extended sense. Then $R = \frac{1}{L}$ (including the cases $L=0$, $L=\infty$).*

PROOF. Immediate. $\square$

EXAMPLE 218. Consider $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ which converges for $|x|<1$. We can also expand around other points. For example,

$$\frac{1}{1-x} = \frac{1}{(5-x)-4} = -\frac{1}{4}\cdot\frac{1}{1-\frac{5-x}{4}} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{5-x}{4}\right)^n = \sum_{n=0}^{\infty}(-4)^{-n-1}(x-5)^n$$

which converges if $\left|\frac{5-x}{4}\right|<1$, that is if $x\in(1,9)$.

EXAMPLE 219. Consider $\sum_{n=0}^{\infty}\frac{1}{4^k}\binom{2k}{k}x^k$. $\binom{2k+1}{k+1}4^{-k-1}/\binom{2k}{k}4^{-k} = \frac{2(2k+1)}{4(k+1)} = \frac{k+\frac{1}{2}}{k+1}\to 1$ so the radius of convergence is 1.

5.4.2. Properties of power series. Fix a power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ with radius of convergence $R>0$ (possibly $R=\infty$). Write $f(x)$ for the sum of the series at a point x where it convergence.

THEOREM 220. *$f(x)$ is continuous on the open interval (x_0-R, x_0+R).*

PROOF. It is enough to show continuity on every proper subinterval (x_0-S, x_0+S) where $S<R$. Accordingly fix S and choose x_1 such that $x_0+S<x_1<x_0+r$. Then the series converges at x_1 so there is N so that if $n\geq N$ $|a_n(x_1-x_0)^n|\leq 1$. Set $\lambda = \frac{S}{x_1-x_0}$; by assumption $0<\lambda<1$.

Truncate the sum at T, that is write

$$f(x) = \sum_{n=0}^{T} a_n(x-x_0)^n + \sum_{n=T+1}^{\infty} a_n(x-x_0)^n .$$

The idea is that the first sum is a polynomial, hence a continuous function, and the second sum is small by the proof of convergence above. In detail, suppose $T\geq N$. Then

$$\begin{aligned}\left|\sum_{n=T+1}^{\infty} a_n(x-x_0)^n\right| &\leq \sum_{n=T+1}^{\infty}|a_n(x_1-x_0)^n|\left|\frac{x-x_0}{x_1-x_0}\right|^n \\ &\leq \sum_{n=T+1}^{\infty} 1\cdot\lambda^n \\ &= \frac{\lambda^{T+1}}{1-\lambda}.\end{aligned}$$

In particular, given $\varepsilon>0$ there is T large enough so that $\frac{\lambda^{T+1}}{1-\lambda}<\frac{\varepsilon}{3}$. Now given $x\in(x_0-S,x_0+S)$ the continuity of the polynomial $\sum_{n=0}^{T} a_n(x-x_0)^n$ at x gives $\delta>0$ so that if $|y-x|<\delta$ then

$\left|\sum_{n=0}^{T} a_n(x-x_0)^n - \sum_{n=0}^{T} a_n(y-x_0)^n\right| < \frac{\varepsilon}{3}$ (we suppose that δ is small enough so that $(x-\delta, x+\delta) \subset (x_0-S, x_0+S)$). Then

$$\begin{aligned} |f(x)-f(y)| &\leq \left|\sum_{n=0}^{T} a_n(x-x_0)^n - \sum_{n=0}^{T} a_n(y-x_0)^n\right| + \left|\sum_{n=T+1}^{\infty} a_n(x-x_0)^n\right| + \left|\sum_{n=T+1}^{\infty} a_n(y-x_0)^n\right| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon\,. \end{aligned}$$

It follows that $\lim_{y\to x} f(y) = f(x)$, that is that f is continuous at x. □

REMARK 221. It is a Theorem of Abel's that if the series converges at an endpoint then the function is continuous at that point as well.

THEOREM 222 (Integrate term-by-term). *The series $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1} = \sum_{n=1}^{\infty} \frac{a_{n-1}}{n}(x-x_0)^{n+1}$ has the same radius of convergence as f and for $x \in (x_0-R, x_0+R)$ we have $\int_{x_0}^{x} f(t)\,\mathrm{d}t = F(x)$.*

PROOF. If $|x-x_0| < R$ then $\left|\frac{a_n}{n+1}(x-x_0)^{n+1}\right| \leq |x-x_0|\,|a_n(x-x_0)^n|$ since $\frac{1}{n+1} \leq 1$. We know $\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$ converges absolutely, so the series $F(x)$ also converges. Conversely, suppose that $\sum_{n=0}^{\infty} \left|\frac{a_n}{n+1}(x_1-x_0)^{n+1}\right|$ converges . Then if $|x-x_0| < |x_1-x_0|$, we have$|a_n(x-x_0)^n| \leq \left|\frac{a_n}{n+1}(x_1-x_0)^n\right|(n+1)\left|\frac{x-x_0}{x_1-x_0}\right|^n$. We know that $\left|\frac{a_n}{n+1}(x_1-x_0)^n\right|$ is bounded, and that $\sum_{n=0}^{\infty}(n+1)\lambda^n$ converges if $|\lambda| < 1$, so $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely. It follows that the radius of convergence of f is at least that of F, so both series have the same radius. Finally, given $x \in (x_0-R, x_0+R)$ write $S = |x-x_0| < R$, let A be such that $\left|a_n\left(x_0+\frac{S+R}{2}\right)^n\right| \leq A$ (the series converges at $x_0+\frac{S+R}{2}$) and write $\lambda = \frac{2S}{S+R} < 1$. Then:

$$\left|f(t) - \sum_{n=0}^{T} a_n(t-x_0)^n\right| \leq \sum_{n=T+1}^{\infty} A\lambda^n \leq \frac{A\lambda^{T+1}}{1-\lambda}\,.$$

It follows that

$$\begin{aligned} \left|\int_{x_0}^{x} f(t)\,\mathrm{d}t - \sum_{n=0}^{T} \frac{a_n}{n+1}x^{n+1}\right| &= \left|\int_{x_0}^{x} f(t)\,\mathrm{d}t - \int_{x_0}^{x}\left(\sum_{n=0}^{T} a_n(t-x_0)^n\right)\mathrm{d}t\right| \\ &\leq \left|\int_{x_0}^{x}\left|f(t) - \sum_{n=0}^{T} a_n(t-x_0)^n\right|\mathrm{d}t\right| \\ &\leq |x-x_0|A\frac{\lambda^{T+1}}{1-\lambda}\,.. \end{aligned}$$

Since $\lim_{n\to\infty}\lambda^n = 0$ we see that $\int_{x_0}^{x} f(t)\,\mathrm{d}t = \lim_{T\to\infty}\sum_{n=0}^{T} \frac{a_n}{n+1}x^{n+1} = F(x)$. □

THEOREM 223 (Differentiation term-by-term). *The series $G(x) = \sum_{n=1}^{\infty} na_nx^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$ has the same radius of convergence as f and for $x \in (x_0-R, x_0+R)$ we have $f'(x) = G(x)$.*

PROOF. Since the series of f is obtained from that of G using term-by-term integration, the previous Theorem shows that both series have the same radius of convergence and that $f(x) = \int_{x_0}^{x} G(t)\,\mathrm{d}t$. That f is differentiable and the formula $f'(x) = G(x)$ now follow the continuity of G in its interval of convergence and from the fundamental theorem of calculus. □

COROLLARY 224. *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ *has a positive radius of convergence. Then* f *is infinitely differentiable in* (x_0-R, x_0+R) *and* $a_n = \frac{f^{(n)}(x_0)}{n!}$.

PROOF. Both claims follow by induction, also using $a_0 = f(0)$. $\square$

COROLLARY 225. *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ *hold for* $x \in (x_0-S, x_0+S)$, $S > 0$.

(1) *Suppose that* $f(x) = g(x)$ *on this interval containing* x_0. *Then* $a_n = b_n$ *for all* n *and* $f = g$.
(2) $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *is the Taylor series of* f.

5.4.3. Operation on power series.

THEOREM 226. *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $g(x) = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ *have positive radii of convergence* R, R' *respectively.*

(1) *Let* $\alpha, \beta \in \mathbb{R}$. *Then* $\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)(x-x_0)^n$ *has radius of convergence at least* $\min\{R, R'\}$ *and in that interval its sum equals* $\alpha f(x) + \beta g(x)$.
(2) $\sum_{n=0}^{\infty} \left(\sum_{k+l} a_k b_l\right)(x-x_0)^n$ *has radius of convergence at least* $\min\{R, R'\}$ *and in that interval its sum equals* $f(x)g(x)$.

PROOF. (1) Theorem on linear combination of series; (2) Theorem on multiplication of absolutely convergent series. $\square$

THEOREM 227 (Composition). *Suppose* $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$, $g(t) = x_0 + \sum_{k=1}^{\infty} b_k(t-t_0)^k$ *have positive radii of convergence* R, S *respectively (note that* $b_0 = x_0$*!). Then* $f(g(t)) = \sum_{m=0}^{\infty} c_m(t-t_0)^m$ *in an interval about* t_0, *where* $c_m = \sum_{n=0}^{m} a_n d_{n,m}$ *and* $d_{n,m}$ *is the coefficient of* $(t-t_0)^m$ *in* $\left(\sum_{k=1}^{m} b_k(t-t_0)^k\right)^m$.

COROLLARY 228. *If we wish to expand* $f(g(t))$ *to* m*th order it suffices to truncate* f, g *to* m*th order before the substitution.*

Math 121: In-class worksheet

EXAMPLE 229. Series to memorize:

(1) $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$
(2) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$
(3) $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$
(4) $\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$
(5) $\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

EXERCISE 230. Sum the following series

(1) $\sum_{n=1}^{\infty} \frac{x^n}{n+3}$
(2) $\sum_{n=1}^{\infty} \frac{x^n}{n^2+3n}$
(3) $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+2)!} \frac{1}{2!\cdot 2} - \frac{1}{4!\cdot 2^3} + \frac{1}{6!\cdot 2^5} - \frac{1}{8!\cdot 2^7} + \cdots$

EXERCISE 231. Expand the following functions to fifth order about zero

(1) $e^{\sin x}$
(2) $\frac{\sin(x^2)-(\arctan x)^2}{x^4}$
(3) $e^{2x}\cos(x^2)$

Math 121: Example Solution

PROBLEM 232. Let X be a random variable with normal (Gaussian) distribution. Let $\Phi(t)$ be the probability that X is within t standard deviations from its mean. Calculate $\Phi(t)$ to within $\frac{1}{1000}$ where

(1) $t = 1$
(2) $t = 2$

Solution:

CHAPTER 6

Examples from Physics

[313 PS2 for an example of Taylor expansion]

6.1. van der Waals equation

Ideal gas:

$$P = \rho k T$$

where ρ is the particle density (particles per unit volume), k is the Boltzmann constant, and T is the temperature.

Better version:

$$P + a' \rho^2 = \frac{\rho k T}{1 - \rho b'}.$$

Here b' is the volume of a single particle, and a' is a measure of the attraction between particles. Say we want to understand how b affects the density. How do we do this? Taylor expansion!

$$P = (\rho k T) \left[1 + (\rho b') + (\rho b')^2 + \cdots\right] - a' \rho^2$$

6.2. Planck's Law

- Blackbody problem
- Spectral density
- Classical calculation (Rayleigh-Jeans) give the spectral density as

$$B_\lambda(T) d\lambda = 2ckT \frac{\mathrm{d}\lambda}{\lambda^4}$$

or ($\lambda = \frac{c}{\nu}$ so $d\lambda = -\frac{c}{\nu^2}$)

$$B_\nu(T) = \frac{2kT}{c^2} \nu^2 \,\mathrm{d}\nu$$

- Problem: the integral diverges (infrared catastrophe); predicts too much power at small scales.
- Planck: better formula is

$$B_\lambda(T) d\lambda = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k T}} - 1} \mathrm{d}\lambda$$

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1} \mathrm{d}\nu$$

Physical derivation: Consider a specific oscillator in the box, of energy quantum E. The probability that there are n photos of energy E is proportional to

$$e^{-\beta n E}.$$

We first find the normalizing constant, giving the probability exactly

$$\left(1-e^{-\beta E}\right)e^{-\beta n E}.$$

The expected energy of a specific oscillator is therefore

$$\begin{aligned}
\left(1-e^{-\beta E}\right)\sum_{n=0}^{\infty} nEe^{-\beta n E} &= -\left(1-e^{-\beta E}\right)\frac{\partial}{\partial \beta}\sum_{n=0}^{\infty} e^{-\beta n E} \\
&= \left(1-e^{-\beta E}\right)\frac{Ee^{-\beta E}}{(1-e^{-\beta E})^2} = \\
&= \frac{E}{e^{\beta E}-1}
\end{aligned}$$

We now suppose with Planck that $E = h\nu$; in the final expression for the density ν counts polarizations and $\nu^2\,\mathrm{d}\nu$ is the density of states.

$$\frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}$$

PROBLEM 233. Find peak (at $\frac{2.82}{\beta}$), total intensity ($\propto T^4$)

Bibliography