Math 121: Problem set 9 (due 20/3/12)

Practice problems (not for submission!)

Sections 9.1: All problems.

Limits of Sequences

1. Choose an arbitrary real number a_0 and consider the sequence given by $a_{n+1} = \cos(a_n)$.
   DO IT! Input a number into your calculator and repeatedly press the cosine button. Now retry with a different starting point.
   SUPP Show that there a unique real number ξ so that $\cos\xi = \xi$, and that $0 \le \xi \le 1$.
   (b) Show that for $n \ge 2$, $0 \le a_n \le 1$.
   (c) For $n \ge 2$ show that $|a_{n+1} - \xi| \le \sin(1)\,|a_n - \xi|$.
   Hint on other side.
   (d) Show that $|a_{n+2} - \xi| \le (\sin(1))^n\,|a_2 - \xi|$.
   (e) Show that $\lim_{n\to\infty} a_n = \xi$.

2. (Final exam 2009) Define a sequence by a_n by $a_0 = 3$ and $a_{n+1} = \frac{2}{3}a_n + \frac{4}{3a_n}$.
   (a) Show that $\frac{4}{3} \le a_n \le 3$ for all n.
   [Part (b) in the exam: Show that the sequence converges and evaluate the limit]
   (b) Suppose the $L = \lim_{n\to\infty} a_n$ existed. Find its value.
   (c) Using the value from part (b), show that $\frac{1}{6} = \frac{2}{3} - \frac{1}{2} < \frac{a_{n+1}-L}{a_n-L} \le \frac{2}{3}$ for all n.
   Hint on other side.
   (d) Show that $0 < a_n - L \le \left(\frac{2}{3}\right)^n$ for all n; conclude that $\lim_{n\to\infty} a_n$ exists and equals L.
   Hint: You can either use the definition of the limit or just quote a theorem.

3. Evaluate the following limits:
   (a) $\lim_{n\to\infty} \sqrt[n]{2^n + 3^n}$.
   Hint on other side.
   (b) $\lim_{n\to\infty} \left(\sum_{i=1}^{k} b_i^n\right)^{1/n}$ where $b_1, \ldots, b_k$ are fixed positive real numbers.

4. Let $a_0 = 1$ and $b_0 = 2$. Recusively set $a_{n+1} = \sqrt{a_n b_n}$, $b_{n+1} = \frac{a_n+b_n}{2}$.
   DO IT! Use a calculator or computer to find the first few values of the sequence. At what point does $b_n - a_n = 0$ hold within the precision of your calculator?
   (a) Show that $b_{n+1} - a_{n+1} = \frac{(a_n-b_n)^2}{2(\sqrt{a_n}+\sqrt{b_n})^2} \le \frac{(a_n-b_n)^2}{8}$.
   (b) Show that $b_n - a_n \le 2^{-2^n}$ for $n \ge 1$. Find N so that if $n \ge N$ then $b_n - a_n \le 10^{-100}$.
   RMK Convergence is very fast here.

Hint for 1(c): $\sin(1)$ is an upper bound for the derivative of $\cos x$ in $[0,1]$.
Hint for 2(c): subtract $L = \frac{2}{3}L + \frac{4}{3L}$ from the recursion relation and use $\frac{1}{a_n L} < \frac{1}{2}$ (why?)
Hint for 3(a): $3^n \leq 2^n + 3^n \leq 2 \cdot 3^n$; now use squeeze.

Exam practice: A continued fraction

A. Define a sequence by $a_0 = 1$ and $a_{n+1} = \frac{1}{1+a_n}$.
   (a) [Low hanging fruit I] Show that $0 < a_n < 1$ for all $n \geq 1$.
   (b) [Low hanging fruit II] Suppose $L = \lim_{n\to\infty} a_n$ exists; find it!
   (c) Show that $a_{n+1} - a_n = -\frac{a_n - a_{n-1}}{(1+a_n)(1+a_{n-1})}$ for all n.
   (d) Show that $a_1 < a_3 < a_5 < \cdots < a_{2k-1} < a_{2k+1} < \cdots < a_{2k} < a_{2k-2} < \cdots < a_4 < a_2 < a_0$. *Hint*: induction.
   (e) Show that $A = \lim_{k\to\infty} a_{2k+1}$ and $B = \lim_{k\to\infty} a_{2k}$ exist and that $A \leq B$.
   (*f) Show that $|a_{n+1} - a_n| \leq \frac{|a_n - a_{n-1}|}{\left(1+\frac{1}{2}\right)^2}$ if $n \geq 2$. Use that to show that $\lim_{n\to\infty} |a_{n+1} - a_n| = 0$ and hence that $A = B$, so that $\lim_{n\to\infty} a_n$ exists.
   RMK The limit you have found is normally written as $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}}}$

Supplementary problem: Defining raising to powers

B. Let $x \in \mathbb{R}$. Define $x^0 = 1$ and $x^{n+1} = x \cdot x^n$ for all integral $n \geq 0$.
   (a) Show that $x^{a+b} = x^a x^b$ for all natural numbers a, b.
   (b) Show that $(x^a)^b$ for all natural numbers a, b.
   (c) Show that $(xy)^a = x^a y^a$ for natural numbers a.
   (d) Show that if $0 < x < y$ and a is a natural number then $0 < x^a < y^a$. Conclude that if $x \neq 0$ then $x^a \neq 0$ for all a.

C. Let $x \in \mathbb{R}$ be non-zero. If n is a negative integer set $x^n = \frac{1}{x^n}$.
   (a) Show that $x^{a+b} = x^a x^b$ for all $a, b \in \mathbb{Z}$.
   (b) Show that $(x^a)^b$ for all $a, b \in \mathbb{Z}$.
   (c) Show that $(xy)^a = x^a y^a$ for all $a \in \mathbb{Z}$.
   (d) Show that if $0 < x < y$ and a is a negative integer then $x^a > y^a > 0$.

D. Fix $n \geq 1$ and consider the function $f(x) = x^n$ for $x \geq 0$.
   (a) Show that f is continuous on its domain.
   (b) Show that f is strictly monotone.
   (c) Show that for any $y \geq 0$ there are $0 \leq x_1 \leq x_2$ so that $f(x_1) \leq y \leq f(x_2)$.
   (d) Conclude that every non-negative real has a unique nth root.

E. For a rational number $\frac{p}{q}$ where p, q are integers with q positive and for $x \geq 0$ set $x^{p/q} = (x^p)^{1/q}$ where x^p was defined in part C and qth roots are defined as in part D. Show that properies (a)-(d) of problems B,C hold where a, b range over the rationals.

F. Let $b \geq 0$ be a fixed real number, and consider the function $g(r) = b^r$ for r rational.
   (a) Show that g is monotone.
   (**b) Given $\varepsilon > 0$ there is $\delta > 0$ so that if $|r| < \delta$ then $|b^r - 1| < \varepsilon$.
   (*c) Conclude that the function g has no "jumps" and is hence extends to a continuous function defined for every real number.
   (d) Show that the exponentiation b^x as defined in part (c) satisfies the properties of C(a)-C(d).
   (e) Show that $b \mapsto b^x$ is continuous for fixed x.