Math 121: Problem set 6 (due 20/2/12)

Practice problems (not for submission!)

Sections 7.1-7.4

Length, Area, Volume

1. Find the volume of the solid obtained by rotating a disc around one of its tangent lines.
   (a) Slicing by cylinderical shells centered at the axis of rotation.
   (b) Slicing by planes perpendicular to the axis of rotation.

2. A marble sculpture of height H sits on a circular stone plinth (base). The volume of the part of the sculpture lying at height at most z above the plinth (that is, between the plane of the plinth and a parallel plane at height z) is $(4H-3z)z^3$. What is the cross-sectional area of the sculpture at height z?

3. Let f be continuous and positive on the interval $[a,b]$, and let S be the surface obtained by revolving the graph of $y=f(x)$ about the x-axis.
   (a) Argue that the part of the surface lying between the planes $x=x_i$ and $x=x_i+\Delta x$ has approximate area $2\pi f(x_i)\cdot\sqrt{(\Delta x)^2+(\Delta y)^2}$ where $\Delta y = f(x_i+\Delta x)-f(x_i)$.
   (b) Argue that the surface S has area
   $$2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,\mathrm{d}x\,.$$

4. Let R be the infinite horn obtained by revolving the curve $y=\frac{1}{x^p}$, $x\geq 1$, around the x axis.
   (a) Find the volume of the solid bounded by the horn and the plane $x\geq 1$ (for which p is the volume finite?)
   (b) Find the center-of-mass of the solid, if it exists.
   (c) Find the surface area of the horn (for which p is the area finite)?
   SUPP For some values of p the horn has finite volume but infinite surface area. In other words, we can fill the horn with a finite amount of paint yet no finite amount of paint suffices to coat its surface. How is this possible?

5. A ball of radius R meters is filled with a heavy gas. This gas sinks to the bottom so that it has density $A(4R-z)\frac{\mathrm{kg}}{\mathrm{m}^3}$ at height z above the bottom of the ball (A is a constant). Find the mass of the gas and the location of its center of mass.

Supplementary problems

A (On the FTC I) Let g be continuous, and let f agree with g at all points but x_0. Show that $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is differentiable for all x but that $F'(x_0) \neq f(x_0)$.

B. (on the FTC II) Let $h(x) = \sin\frac{1}{x}$ for $x \neq 0$.

(a) Show that $H(x) = \int_0^x h(t)\,\mathrm{d}t$ exists for all x (note that the value of $h(0)$ is irrelevant for this).

(b) Show that $H(x)$ is differentiable for all x, and define $h(0)$ so that $H'(x) = h(x)$ for all x.

C. The *gravitational potential* at a distance r from a mass m is $\frac{Gm}{r}$ where G is Newton's constant.

(a) Find the gravitational potential at a distance z above the center of a flat disc of radius R and mass ρ per unit area.

(b) Find the gravitational potential at a distance $r \geq R$ from the center of a ball of radius R and mass ρ per unit volume.
Hint: Let the z axis run through the center of the ball and the point at which the potential is being calculated; slice in planes perpendicular to the z-axis and apply (a).

(c) Obtain *Newton's formula*: the gravitational potential at a point outside a ball of constant density is $\frac{GM}{r}$ where M is the mass of the ball and r is the distance from the point to the center of the ball.

(*d) Show that (c) holds even if the density