Math 121: Problem set 4 (due 3/2/12)

Practice problems (not for submission!)

Section 6.5.

Integration

1. Evaluate
   (a) $\int \frac{e^x+1}{e^{3x}+5e^{2x}+4e^x+20}\,\mathrm{d}x$.
   (b) $\int \frac{\mathrm{d}x}{x^3\sqrt{1+x^2}}$.

2. Let f be a real valued function defined on $\mathbb{R}$. Suppose that f is identically zero outside the interval $[a,b]$, and that is *infinitely differentiable* (also called *smooth*), that is that $f^{(k)}$ exists for all k. We will study the integrals $A(f;\lambda) = \int_{-\infty}^{+\infty} f(x)\cos(\lambda x)\,\mathrm{d}x = \int_a^b f(x)\cos(\lambda x)\,\mathrm{d}x$ and $B(f;\lambda) = \int_{-\infty}^{+\infty} f(x)\sin(\lambda x)\,\mathrm{d}x = \int_a^b f(x)\sin(\lambda x)\,\mathrm{d}x$ for large λ. Below we always assume $\lambda \neq 0$.
   SUPP Show by induction that for all $k \geq 0$ $f^{(k)}$ is continuous and vanishes outside $[a,b]$ (and therefore also at a,b). Conclude that there are constants M_k so that $\left|f^{(k)}(x)\right| \leq M_k$ for all x,k.
   (b) Show that $|A(f;\lambda)|, |B(f;\lambda)| \leq (b-a)M_0$.
   (c) Show that $A(f;\lambda) = -\frac{1}{\lambda}B(f';\lambda)$ and that $B(f;\lambda) = \frac{1}{\lambda}A(f';\lambda)$. Conclude that $|A(f;\lambda)|, |B(f;\lambda)| \leq \frac{(b-a)M_1}{|\lambda|}$.
   Hint: Integration by parts.
   (d) Show that $|A(f;\lambda)|, |B(f;\lambda)| \leq \frac{(b-a)M_k}{|\lambda|^k}$ holds for all k.
   RMK The integrals A,B (considered as functions of λ) are (together) called the *Fourier Transform* of f [An "integral transform" is an operation that converts a function of x to a function of λ by integrating f against a function of both x and λ, here against $\cos(x\lambda), \sin(x\lambda)$]. You have shown that f being differentiable translates to *rapid decay* of its Fourier Transform.

Asymptotics and improper integrals

3. Show that $\frac{1}{\sqrt{x}} \sim_0 \frac{1+x}{\sqrt{x}} \sim_0 \frac{1}{\sqrt{\sin x}} \sim_0 \frac{1}{1-e^{-\sqrt{x}}}$.

4. Decide whether the following integrals converge without evaluating them.
   (a) $\int_0^1 \frac{\mathrm{d}x}{x(1-x)}$.
   (b) $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)}}$
   (c) $\int_0^1 \frac{\mathrm{d}x}{(x(1-x))^{1/3}}$.
   (d) $\int_0^\infty \frac{1-\cos x}{x^3}\,\mathrm{d}x$.
   (*e) $\int_{10}^\infty \frac{\mathrm{d}x}{x^p \log x}$ (your answer will depend on p!)

5. Evaluate the integral $\int_0^1 \frac{\mathrm{d}x}{\sqrt{x(1-x)}}$.
   Hint: Shift the function so its axis of symmetry is the y-axis.

6. Euler's Gamma function is the function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,\mathrm{d}x$.
   (a) Show that the integral converges for $z > 0$.
   (b) Use integration by parts to show $\Gamma(z+1) = z\Gamma(z)$ in the region of convergence.
   (c) Show that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.
   Hint: Induction.

Supplementary problems

A. Fun with arctan.
   (a) Show that for $a \neq 0$, $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a})$.
   Hint: We basically did this in class.
   (*b) Show that for any x,y $\lim_{a\to 0} \frac{\arctan(\frac{x}{a})-\arctan(\frac{y}{a})}{a} = \frac{1}{y} - \frac{1}{x}$ (note that the RHS is $\int_x^y \frac{1}{t^2}dt$).

B. Properties of $\sim_a$:
   (a) Show that $f \sim_a f$ for all f, that if $f \sim_a g$ and $g \sim_a h$ then $g \sim_a f$ and $f \sim_a h$.
   (b) Show that $f \sim_a g$ and $k \sim_a l$ implies $fk \sim_a gl$ and $\frac{f}{k} \sim_a \frac{g}{l}$. Why are you not dividing by zero?

Supplementary problems – Polynomials and Partial fractions

Let F be a field (see Definition 41). Write $F[x]$ for the set of polynomials over F (if $F = \mathbb{R}$ then $F[x]$ includes elements like 0, x, $\pi x^2 + (e^2-2)x - 7$). The *degree* of a polynomial is the degree of its heighest monomial.

C. (The division algorithm) Let $f,g \in F[x]$ be polynomials with $g \neq 0$.
   (a) Suppose that $\deg f \geq \deg g$. Show that there is a constant $c \in F$ so that the polynomial $f - (cx^{\deg f - \deg g})g$ has degree strictly smaller than that of f.
   (b) Show that there are $q,r \in F[x]$ so that $f = qg + r$ and such that $\deg r < \deg g$.
   Hint: Induction on $\deg f$ like the proof of partial fractions in class.
   (c) Show that the q,r in part (b) are unique: that if $qg + r = q'g + r'$ where $\deg r' < \deg g$ as well then $q = q'$ and $r = r'$.
   Hint: Show that g would divide $r - r'$.

D. (Partial fractions in general)
   (a) Show that the proof of Proposition 101 (and its preceeding Lemma) holds over any field.
   (b) The "Fundamental Theorem of Algebra" states that every polynomial $Q \in \mathbb{C}[x]$ of positive degree has a complex root. Deduce that over $\mathbb{C}$ every ratio $\frac{P}{Q}$ can be expressed as the sum of a polynomial and terms of the form $\frac{C_{i,j}}{(x-a_i)^j}$.

E. Application
   (a) Find the complex partial fraction expansions of $\frac{1}{1+x^2}$, $\frac{1}{x^3-1}$.
   (b) Show that $\int \frac{\mathrm{d}x}{1+x^2} = \frac{i}{2}\log\frac{x+i}{x-i} + C$.
   (**c) Show that (at least for x real) $\frac{i}{2}\log\frac{x+i}{x-i} + C = \arctan x + C$ for an appropriate branch of the logarithm.