Math 121: Problem set 2 (due 20/1/12)

Foundations

1. (Density of rationals and irrationals).
   (a) Let $x, T \in \mathbb{R}$ with $T > 0$. Let n be the largest integer so that $n \leq Tx$. Show that $\frac{n}{T} \leq x \leq \frac{n+1}{T}$, and in particular that $\left|x - \frac{n}{T}\right| \leq \frac{1}{T}$.
   (b) Show that any interval of the form $[a, b]$ contains a rational number.
   Hint: In part (a) take $x = a$ and T to be a large integer.
   (c) Show that any interval of the form $[a, b]$ contains an irrational number.
   Hint: Choose T differently.

Calculus

2. Let f be integrable on $[a, b]$. We show that $-f$ is integrable on this interval and that $\int_a^b (-f)(x)dx = -\int_a^b f(x)dx$.
   (a) For any paritition P express $L(-f;P), U(-f;P)$ in terms of $L(f;P), U(f;P)$.
   (b) Let P be such that $L(f;P), U(f;P)$ are within ε of $I = \int_a^b f(x)dx$. Show that $L(-f;P), U(-f;P)$ are within ε of $-I$.

3. Let R be the bounded region bounded by the graphs of $f(x) = \log(x)$, $g(x) = x^2 - 2$ (formally, $R = \left\{(x, y) \mid x > 0; x^2 - 2 \leq y \leq \log x\right\}$).
   – You may want to draw yourself a picture of this region. Note that log denotes the natural logarithm.
   (a) Show that the two graphs intersect at exactly two points. Call $a < b$ the x-coordinates of those points.
   (b) Let $a < x_{i-1} < x_i < b$. In this part we consider the intersection of the region R with the strip determined by the interval $[x_{i-1}, x_i]$. Write $M_i(f) = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$, and similarly $m_i(g), M_i(g)$ for the quantities for g. Using only the numbers $x_{i-1}, x_i, m_i(f), M_i(f), m_i(g), M_i(g)$ describe two rectangles in the plane, one contained in the intersection and one containing the intersection.
   Hint: The height of one of the rectangles is $M_i(f) - m_i(g)$.
   (c) Show that for any partition P of $[a, b]$, the area of R satisfies $L(f;P) - U(g;P) \leq \text{Area}(R) \leq U(f;P) - L(g;P)$.
   Hint: Sum your results from part (b).
   (d) Given that f, g are integrable on $[a, b]$ show that $\text{Area}(R) = \int_a^b f(x)dx - \int_a^b g(x)dx$.

4. Find a function f so that $\int_x^{10} f(t)dt = 5 - f(x)$ for all $x \in \mathbb{R}$.
   Hint: Differentiate with respect to x

Supplementary problem – an integrable function

A. The *Riemann function* is defined by

$$R(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q},\, p,q \in \mathbb{Z} \text{ relatively prime with } q \geq 1 \\ 0 & x \text{ irrational} \end{cases}.$$

(a) Show that $R(x)$ is continuous at $x_0 \in \mathbb{R}$ iff x_0 is irrational.

(b) Given $\varepsilon > 0$ construct a partition P of $[0,1]$ for which $U(R;P) \leq \varepsilon$.
Hint: Around each rational $\frac{p}{q}$ with $1 \leq q \leq T$ take a small interval of length δ. How big does R get outside those intervals?

(c) Show that $R(x)$ is integrable on every interval and that its integral is zero.

Supplementary problems – the natural numbers

A. Call a subset $A \subset \mathbb{R}$ *inductive* if $0 \in \mathbb{R}$ and if $x \in \mathbb{R}$ implies $x+1 \in \mathbb{R}$.

(a) Show that $\mathbb{R}$ itself is inductive. are all inductive.

(b) Show that $[0,\infty)$, $\mathbb{Q} \cap [0,\infty)$, and $\{0\} \cup [1,\infty)$ are all inductive.

(c) Let $A \subset \mathbb{R}$ be inductive, and suppose that M is an upper bound for A. Show that $M-1$ is also an upper bound.
Hint: For $x \in A$ show that $x+1 \leq M$.

(d) Show that no inductive set is bounded above.

B. The *set of natural numbers* is by definition $\mathbb{N} \stackrel{\text{def}}{=} \bigcap \{A \mid A \text{ is inductive}\} = \{x \in \mathbb{R} \mid x$ belongs to every inductive

(a) Show that $0 \in \mathbb{N}$, $1 \in \mathbb{N}$, $2 \in \mathbb{N}$.

(b) Show that every element of $\mathbb{N}$ is non-negative, and that there is no $n \in \mathbb{N}$ so that $0 < n < 1$..
Hint: A(a), A(b).

(c) Show that $\mathbb{N}$ is inductive. Conclude from 1(c) that $\mathbb{R}$ has the *archimedean property*: for every $M \in \mathbb{R}$there is $n \in \mathbb{N}$ such that $n \geq M$.

(d) Conclude that for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ so that $\frac{1}{n} < \varepsilon$.

(e) Show that $\mathbb{N}$ is the smallest inductive set: that if A is inductive then $\mathbb{N} \subset A$.

REMARK. B(c) is the principle of induction!

C. Properties of the natural numbers

(a) Show that $\mathbb{N}$ is closed under addition.
Hint: Show that $\{n \in \mathbb{N} \mid \text{for all } m \in \mathbb{N},\, m+n \in \mathbb{N}\}$ is inductive.

(b) Show that $\mathbb{N}$ is closed under multiplication.
Hint: Show that $\{n \in \mathbb{N} \mid \text{for all } m \in \mathbb{N},\, mn \in \mathbb{N}\}$ is inductive.

(c) Show that $\{n \in \mathbb{N} \mid n = 0 \text{ or } n-1 \in \mathbb{N}\}$ is inductive. Conclude that if $n \in \mathbb{N}_{\geq 1}$ then $n-1 \in \mathbb{N}$.

(d) Show that if $n,m \in \mathbb{N}$ and $n \geq m$ then $n-m \in \mathbb{N}$.

(e) Show that $\mathbb{N}$ is *discrete*: if $n \in \mathbb{N}$ then $(n-1,n+1) \cap \mathbb{N} = \{n\}$.
Hint: Deduce this from B(b) and C(d).