Math 121: Problem set 1 (due 13/1/12)

Submit solutions to numbered problems only. Practice problems are selected from the text for solving at home. Lettered problems as well as ones labelled SUPP are supplementary, and not for submission (they cover additional topics or are more abstract). RMK indicates a remark, not an exercise.

Practice problems (not for submission!)

Section 5.1: problems 1-39
Section 5.2: all problems
Section 5.3: 3-5, 7-17
Section 5.4: all problems.

Induction and the Σ notation

1. Write the following sums using a Σ:
   (a) $\sqrt{4}+\sqrt{5}+\sqrt{6}+\cdots+\sqrt{37}$;
   (b) $1-2+4-8+16-32+\cdots$ so that there are n terms.
   (c) $\frac{0}{1}+\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$ so that there are n terms.

2. (Summation formulas)
   (a) Evaluate the sums from 1(b),1(c) for $n=10$.
   (b) Show by induction on n that for $q \neq 1$, $\sum_{k=0}^{n-1} q^k = \frac{q^n-1}{q-1}$ (for $n=0$ interpret the LHS as the empty sum).
   Hint: Add q^n to both sides.
   (c) Show that $\sum_{k=0}^{n-1} kq^k = \frac{q}{(q-1)^2}\left(nq^{n-1}(q-1)-(q^n-1)\right) =$
   Hint: Use a derivative.
   (d) Find n so that the sum in 1(c) is within 10^{-100} of 2.
   Hint: Use $\frac{(1/2)}{(1-(1/2))^2} = 2$ in 2(c).

Riemann sums

3. Interpret the following expressions as Riemann sums. Give the interval, function, partition, and chosen points.
   (a) $\frac{1}{n}\sum_{i=0}^{n-1}\sin\left(\frac{i}{n}\right)$;
   (b) $\frac{1}{n}\sum_{i=1}^{n}\sin\left(\frac{i}{n}\right)$;
   (c) $\sum_{i=1}^{n}\frac{n}{i^2+n^2}$.

4. In this problem we compare the Riemann sums associated to different partitions. These inequalities were crucial in the definition of the Riemann integral given in class. We consider a function f defined on a closed interval $[a,b]$.
   (a) Let P_1, P_2 be two partitions of $[a,b]$ so that every point of P_1 is also a point of P_2. Show that
   $$L(f,P_1) \leq L(f,P_2) \leq U(f,P_2) \leq U(f,P_1)$$
   Hint: Start with the case where P_2 extends P_1 by one point.
   (b) Now let P_1, P_2 be *any* two partitions of $[a,b]$. Use your answer to (a) to show that $L(f;P_1) \leq U(f,P_2)$, in other words that *any* lower Riemann sum is smaller than *any* upper Riemann sum.

Supplementary problems on Induction – The factorial function and binomial coefficients (if you want to practice induction)

Recall that the factorial function is defined by $0! = 1$ and for $n \geq 0$ by $(n+1)! = (n+1) \cdot n!$, that is $n! = \prod_{j=1}^{n} j$. The *binomial coefficients* are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n \geq 0$ we set $\binom{n}{k} = 0$ (for example, $\binom{4}{2} = 6$ while $\binom{2}{4} = 0$).

A. Evaluate $\binom{6}{3}$, $\binom{n}{0}$, $\binom{n}{1}$ for all n.

B. (Integrality)
   (a) Show that for all $n, k \geq 0$ we have $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$.
   Hint: Direct calculation.
   (b) Let $A = \left\{n \in \mathbb{N} \mid \forall k \in \mathbb{N} : \binom{n}{k} \in \mathbb{N}\right\}$. Show that $A = \mathbb{N}$, that is that all binomial coefficients are integers.

C. (The Binomial Theorem) Show by induction on n that for all $x, y \in \mathbb{R}$and all integral $n \geq 0$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$.

Suppelementary Problems – The least upper bound proprety (if you want extra foundational material)

Let $A \subset \mathbb{R}$. We say that $M \in \mathbb{R}$is an *upper bound* for A if for all $a \in A$ one has $a \leq M$. We say that M is the *least upper bound* if M is an upper bound, and for every other upper bound M' we have $M \leq M'$. If A has an upper bound we say that it is *bounded above.* The notion of *lower bound* and *bounded below* are defined analogously. We say that A is *bounded* if its both bounded above and bounded below.

A. For each of the following sets determine whether it is bounded above, and whether it is bounded below. Give lower or upper bounds or prove that they don't exist as appropriate. *You may try to find optimal bounds, but the problem isn't asking for those.*
   (a) $\left\{x \in \mathbb{R} \mid x^2 < 2\right\}$.
   (a) $\left\{x \in \mathbb{R} \mid x^3 - x \geq 5\right\}$;
   (b) $\left\{x^2 - 6x + 1 \mid x \in \mathbb{R}\right\}$;
   (c) $\{\cos(x) \mid x \in \mathbb{R}\}$.

B. Simple properties.
   (a) Let $A \subset \mathbb{R}$ be non-empty and let M be an upper bound for A. Let $M' > M$. Show that M' is also an upper bound.
   (b) Assume that A is bounded above. Show that the set $\{M \in \mathbb{R} \mid M \text{ is an upper bound for } A\}$ is bounded below.

We now show that $\mathbb{R}$ has the *least upper bound property*: If $A \subset \mathbb{R}$ is non-empty and has an upper bound, it has a least upper bound. The proof should remind you of the proof of the Intermediate Value Theorem for continuous functions.

C. Let $A \subset \mathbb{R}$ be non-empty, and let M be an upper bound. We define two sequences $\{a_n\}_{n=0}^{\infty} \subset A$, $\{M_n\}_{n=0}^{\infty} \subset \mathbb{R}$ as follows. First, let a_0 be any element of A, and let M_0 be any upper bound for A. Next, given $a_n \in A$ and an upper bound M_n for A, consider $\frac{a_n+M}{2}$. If this is an upper bound for A set $M_{n+1} = \frac{a_n+M}{2}$ and $a_{n+1} = a_n$. Otherwise, set $M_{n+1} = M_n$ and let a_{n+1} be any element of A larger than $\frac{M_n+a_n}{2}$.

(a) Show by induction that for all n, $a_n \in A$ while M_n are upper bounds for A. Also show that $a_{n+1} \geq a_n$ and that $M_{n+1} \leq M_n$.

(b) Conclude that for all k, l we have $a_k \leq M_l$.

(c) Show by induction that $0 \leq M_n - a_n \leq 2^{-n}(M_0 - a_0)$.

(d) Use the completeness axiom from class to get L which is larger than all the a_n but smaller than all the M_n.

(d) Show that L is a *least upper bound* for A.
Hint: You need to check that L is an upper bound but that no smaller number is.

Notation: If A is non-empty and bounded above write $\sup A$ (read: "supremum of A") for its least upper bound.

D. Applications

(a) Let $A = \left\{x \in \mathbb{R} \mid x^2 < 2\right\}$. Show that $(\sup A)^2 = 2$ and conclude that $\sqrt{2}$ is a real number.

(b) (The Intermediate Value Theorem) Let $f(x)$ be continuous on $[a,b]$, and let $f(a) < t < f(b)$. Let $A = \{x \in [a,b] \mid f(x) < t\}$. Show that A is non-empty and bounded above, that $\sup A \in [a,b]$, and that $f(\sup A) = t$.