

Math 121: Sample Problem Set I

Example problems

- Write the following sum using a Σ : $1 + 2 + 3 + 4 + 5 + 6 + 8$.
- Show that for all $n \geq 0$ we have $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.
- In economic theory, money held in the present is more valuable than money that will be available in the future. This difference in value is measured using the *discount rate*: the value **today** (“present value”) of the promise “you will get a dollar a year from now” is written as $\frac{1}{1+r}$ dollars for some $r > 0$.
 - Assuming the discount rate is constant (independent of time), show by induction that the present value of a dollar k years from now is $\frac{1}{(1+r)^k}$ dollars.
 - You have just won the state lottery! You can choose to take a “lump sum” of \$12.5M or 25 annual payments of \$1M each, starting now. Correct to two significant decimal digits, at what discount rate would you be indifferent to the two choices?
- Find n so that $2^n > e^{300}$.
- Find f so that $f'(x) = 2f(x) + 3$.

Sample Solutions

- $\sum_{i=1}^6 i$.
 - Note: No justification is needed for such a simple problem.
- We prove the statement by induction on n . For $n = 0$ the sum is empty, and the RHS evaluates to 1. Assume now that $\sum_{i=1}^n i = \frac{n(n+1)}{2}$. Adding $(n+1)$ to both sides we find:
$$\sum_{i=1}^{n+1} i = \sum_{i=1}^n i + (n+1) = \frac{n(n+1)}{2} + (n+1) = \left[\frac{n}{2} + 1\right] (n+1) = \frac{(n+1)(n+2)}{2} = \frac{(n+1)((n+1)+1)}{2}.$$
 - Note 1: There are other ways of justifying the formula. For example, instead of the sentence “Adding ...” one can say “the second inequality follows from the induction hypothesis”, or connect this statement to the equal sign by an arrow, or choose yet another way to show this.
 - Note 2: Simply writing the chain of formulas is not enough.
- A dollar 0 years from now is worth one of today’s dollars. From the point of view of k years from now, a dollar $k+1$ years from now is worth $\frac{1}{1+r}$ dollars (that dollar would come one year in the future). By induction, 1 dollar k years from now is worth $\frac{1}{(1+r)^k}$ of today’s dollars. So $\frac{1}{1+r}$ dollars k years from now are worth $\frac{1}{(1+r)^k} \cdot \frac{1}{1+r}$ of today’s dollars, as claimed.
 - By part (a), 25 equal annual payments of \$1M each are worth

$$\sum_{k=0}^{24} \frac{(\$1M)}{(1+r)^k} = (\$1M) \sum_{k=0}^{24} \frac{1}{(1+r)^k}$$

of today's dollars. Using the formula for summing a geometric series this equals

$$(\$1\text{M}) = \frac{1 - \left(\frac{1}{1+r}\right)^{25}}{1 - \frac{1}{1+r}} = (\$1\text{M}) \frac{1+r}{r} \left[1 - \left(\frac{1}{1+r}\right)^{25}\right].$$

The alternative is to be paid \$12.5M today. Using a calculator we find that $\frac{1+r}{r} \left[1 - \left(\frac{1}{1+r}\right)^{25}\right] = 12.5$ for $r \approx 6.8\%$.

- Note: no justification for results obtained from a calculator. Exams will not feature such problems.
 - Note: *interpreting* this result goes beyond the mathematics – it goes to the question of whether your personal discount rate is greater or less than 6.8%.
4. We have $2^2 = 4 > e$, so $2^{600} = (2^2)^{300} \geq e^{300}$.
- Note that this kind of problem is generally intended to be solved without the use of calculators.
 - Note that the problem does not ask for the *smallest* n satisfying the condition.
5. Let f be solution. Then $g(x) = f(x)e^{-2x}$ is a differentiable function and by the product rule we have $g'(x) = (f(x)e^{-2x})' = f'(x)e^{-2x} - 2f(x)e^{-2x} = (f'(x) - 2f(x))e^{-2x}$. From the given equation we know that $f'(x) - 2f(x) = 3$ so $g'(x) = 3e^{-2x}$. It follows that $(g(x) + \frac{3}{2}e^{-2x})' = 0$, so this function is constant. We conclude that if f is a solution then there is C so that $g(x) = C - \frac{3}{2}e^{-2x}$ so

$$f(x) = g(x)e^{2x} = Ce^{2x} - \frac{3}{2}.$$

Conversely, each such function is a solution, since

$$\left(Ce^{2x} - \frac{3}{2}\right)' = 2Ce^{2x} = 2\left(Ce^{2x} - \frac{3}{2}\right) + 3.$$

- Note that solving an equation requires **two** steps: (1) showing that if f is a solution then f belongs to some set (here the set of functions of the form $Ce^{2x} - \frac{3}{2}$) and conversely (2) showing that every member of the set actually is a solution.