FOURIER SERIES AND THE POISSON SUMMATION FORMULA (NOTES FOR MATH 613)

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NOTATION

Write S^1 for the circle group $\{z \in \mathbb{C}^\times \mid |z| = 1\}$. We use the number theorists' expoenntial: for $z \in \mathbb{C}$ write $e(z) \stackrel{\text{def}}{=} e^{2\pi i z}$. All group homomorphisms are assumed to be continuous.

For a topological space X write $C(X)$ for the space of $\mathbb{C}$-valued continuous functions on X, $C_c(X)$ for its subspace of functions of compact support. If μ is a Radon measure on X and $1 \le p \le \infty$ write $L^p(\mu)$ for the usual space of [equivalence classes of] p-integrable functions. We sometimes write $L^p(X)$ when the measure is clear (and note that if $L^p(f\mu) = L^p(\mu)$ if f is bounded)

When X is compact, $C(X)$ is complete in the L^∞ norm and (Stone-Weierstrass) a subalgebra $\mathcal{A} \subset C(X)$ is dense iff it separates points, does not have a common zero, and is closed under conjugation.

On a manifold X write $C^j(X)$ for the space of functions differentiable j times with continuous derivatives of order j, $C^\infty(X) = \cap_j C^j(X)$, and $C_c^\infty(X) = C^\infty(X) \cap C_c(X)$.

On $\mathbb{R}^n$ say f is of *rapid decay* if $f(x)(1 + \|x\|)^N$ is bounded for all N, and say $f \in C^\infty(\mathbb{R}^n)$ is of *Schwartz class* if f and all its derivatives are of rapid decay. Write $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz class.

1. LATTICES AND DUAL LATTICES

Let V be an finite-dimensional real vector space.

Exercise 1. Let $\Lambda < V$ be an (abstract) subgroup. Then:

(1) Λ is discrete iff it is of the form $\bigoplus_{i=1}^k \mathbb{Z}\underline{v}_i$ where $\{\underline{v}_i\}_{i=1}^k \subset V$ are linearly independent.
(2) Λ is discrete and V/Λ is compact iff $k = \dim V$, that is if Λ is the $\mathbb{Z}$-span of a basis. In this case we call Λ a *lattice*.
(3) When $\Lambda < V$ is a lattice we have an isomorphism $V/\Lambda \simeq (S^1)^n = \mathbb{T}^n$ where $n = \dim V$.

Fix a lattice $\Lambda < V$, and write $\mathbb{T}$ for the torus V/Λ.

Exercise 2. Let $V^* = \mathrm{Hom}_{\mathbb{R}}(V;\mathbb{R})$ be the dual vector space, and let $\Lambda^* = \mathrm{Hom}_{\mathbb{Z}}(\Lambda;\mathbb{Z})$ be the dual group.

(1) Every $k \in \Lambda^*$ extends uniquely to an element $\varphi \in V^*$.
(2) The extension above induces an embedding $\Lambda^* \hookrightarrow V^*$ whose image is $\{\varphi \in V^* \mid \varphi(\Lambda) \subset \mathbb{Z}\}$ and we identify Λ^* with this image.
(3) Under this identification Λ^* is a lattice in V^*, which we call the *dual lattice*.

Definition. $L^2(\mathbb{T})$ and $L^2(\Lambda^*)$ will denote the L^2-spaces with respect to the Haar probability measure and counting measure, respectively.

Exercise 3 (Functional analysis). (1) Show that $C(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.
(2) Show that $C_c(\Lambda^*)$ is dense in $L^2(\Lambda^*)$.

Definition. For $f \in C(\mathbb{R}^n)$ set $(\Pi_\Lambda f)(x) = \sum_{\lambda\in\Lambda} f(x+\Lambda)$.

Exercise 4. Suppose f decays faster that $(1+|x|)^N$ for N large enough. Show that the series above converges absolutely and that $\Pi_\Lambda f \in C(\mathbb{T})$. If f is j times differentiable and the jth derivative decays fast enough show that $\Pi_\Lambda f \in C^j(\mathbb{T})$. In particular if $f \in \mathcal{S}(\mathbb{R}^n)$ then $\Pi_\Lambda f \in C^\infty(\mathbb{T})$.

Now fix a Haar measure dx on V.

Definition. A *fundamental domain* for V/Λ is an open subset $\mathcal{F} \subset V$ such that:

(1) The translates $\mathcal{F}+\Lambda$ are disjoint, and the translates $\overline{\mathcal{F}}+\Lambda$ cover V.
(2) $\partial\mathcal{F}$ has measure zero.

We also call "fundamental domain" any set between $\mathcal{F}$ and its closure, that is any set whose interior is a fundamental domain and which is contained in the closure of its interior.

Exercise 5 (Fundamental domains). (1) Let $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{\underline{v}_i\}_{i=1}^n$ for a basis $\{\underline{v}_i\}_{i=1}^n \subset V$. Show that $\left\{\sum_{i=1}^n a_i\underline{v}_i \mid \underline{a} \in \left(-\frac{1}{2},\frac{1}{2}\right)\right\}$ and $\left\{\sum_{i=1}^n a_i\underline{v}_i \mid \underline{a} \in [0,1)\right\}$ are fundamental domains.
(2) Suppose that V is an inner product space and let $x_0 \in V$. Show that the *Diriclet domain*

$$\mathcal{F}_{\mathrm{D}} = \{x \in V \mid \forall\lambda \in \Lambda : |x-x_0| \le |(x+\lambda)-x_0|\}$$

is a fundamental domain.

We now connect integration on V and on V/Λ.

Exercise 6 (Integration). (1) Show that there is a unique Haar measure on $\mathbb{T}$ (which we will also denote dx) such that $\int_{\mathbb{T}}(\Pi_\Lambda f)dx = \int_V f dx$.
(2) Let $\mathcal{F}$ be a fundamental domain. Show that for this measure $\mathrm{vol}(\mathbb{T}) = \mathrm{vol}(\mathcal{F})$, and in particular that all fundamental domains have the same finite volume.

Definition. We call $\mathrm{vol}(\mathbb{T})$ the *covolume* of Λ.

2. Fourier Series and Fourier inversion on $\mathbb{R}^n/\Lambda$

Write kx for the pairing between $k \in V^*$ and $x \in V$.

Exercise 7 (Trigonometric polynomials). (1) Let $k \in \Lambda^*$. Show that the function $V \ni x \mapsto e(kx)$ is Λ-invariant and descends to a continuous group homomorphism $e_k : \mathbb{T} \to S^1$.
(2) Show that $k \mapsto e_k$ is an injective group homomorphism $\Lambda^* \hookrightarrow \mathrm{Hom}(\mathbb{T}, S^1)$.
(3) Show that $\{e_k\}_{k\in\Lambda^*} \subset \mathbb{C}(\mathbb{T})$ are linearly independent.
Hint: Evaluate a linear combination $\sum a_k e_k = 0$ of shortest length at two different $x \in \mathbb{T}$.
(4) Let $\mathcal{P}$ be the algebra of continuous functions on $\mathbb{T}$ generated by the e_k. Show that $\mathcal{P}$ is simply the linear span of these characters.
(5) Let $x \in \mathbb{T}$ be non-zero. Show that there exists $k \in \Lambda^*$ such that $e(kx) \neq 1$.
Hint: $\Lambda^{**} = \Lambda$.

(6) Show that $\mathcal{P}$ separates the points of $\mathbb{T}$ and contains 1. By the Stone-Weierstrass Theorem it follows that $\mathcal{P}$ is dense in $C(\mathbb{T})$.

Exercise 8 (Orthogonality of characters)**.** Recall we normalized the Haar measure of $\mathbb{T}$ to be a probability measure.

(1) For $k \in \Lambda^*$ show that $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} e(kx)dx = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$.

(2) Conclude that for $k, \ell \in \Lambda^*$ one has $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} \overline{e(kx)}e(\ell x)dx = \delta_{k\ell}$.

Definition. For $g \in C_c(\Lambda^*)$ set $\check{g}(x) = \sum_{k\in\Lambda^*} g_k(k)e(kx)$.

Exercise 9 (The inverse map)**.** We show that $g \mapsto \check{g}$ extends to an isometric isomorphism $L^2(\Lambda^*) \to L^2(\mathbb{T})$.

(1) (Parseval's identity 1) For $g \in C_c(\Lambda^*)$ show that $\|\check{g}\|_{L^2(\mathbb{T})} = \|g\|_{L^2(\Lambda^*)}$, that is that $\frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} |\check{g}(x)|^2 dx = \sum_{k\in\Lambda^*} |g(k)|^2$.

(2) (Parseval's identity 2) For $g \in L^2(\Lambda^*)$ show that the series $\check{g} = \sum_{k\in\Lambda^*} g_k e_k$ converges absolutely in $L^2(\mathbb{T})$ and that the resulting map $g \to \check{g}$ is an isometric embedding $L^2(\Lambda^*) \to L^2(\mathbb{T})$. Show that the image is a closed subspace.

- Observe that $\check{g} \in L^2(\mathbb{T})$ is only an equivalence class of functions. In particular the statement $\check{g}(x) = \sum_k g_k e(kx)$ need not make sense, and the series of real numbers on the right need not converge.

(3) Let $f \in L^2(\mathbb{T})$ be of norm one and orthogonal to the image of this map. Approximating f by a trigonometric polynomial show that $(f, f) = 0$ and derive a contradiction. Conclude that $g \mapsto \check{g}$ is an isometric isomorphism.

(4) (Decay vs smoothness) We now consider the case where g decays polynomially, in that $|g(k)| \leq C(1+|k|)^{-N}$. Show given j for all sufficiently large N if g decays polynomially with exponent N then $\check{g} \in C^j(TT)$ and for a multi-index α with $|\alpha| \leq j$ we have

$$(\partial^\alpha \check{g})(x) = \sum_{k\in\Lambda^*} (2\pi i)^{|\alpha|} k^\alpha g_k e(kx)$$

in the sense that the series on the right converges absolutely to the value on the left.

Definition. For $f \in L^2(\mathbb{T})$ and $k \in \Lambda^*$ set $\hat{f}(k) = \frac{1}{\mathrm{vol}(\mathbb{T})}\int_{\mathbb{T}} f(x)e(-kx)dx$.

Exercise 10 (The direct map)**.** (1) Show that $\left|\hat{f}(k)\right| \leq \|f\|_{L^2(\mathbb{T})}$. Conclude that $\left|\hat{f}(k)\right| \leq \|f\|_{L^\infty(\mathbb{T})}$ also.

(2) For $g \in C_c(\Lambda^*)$ show that $\hat{\check{g}}(k) = g(k)$. Show that the same holds for $g \in L^2(\Lambda^*)$.

(3) Conclude that the map $f \mapsto \hat{f}$ takes values in $L^2(\Lambda^*)$ and is the inverse to the map $g \mapsto \check{g}$.

Exercise 11 (Smooth functions)**.** (1) Integrating by parts, show that for $k \neq 0$ and $f \in C^{2j}(\mathbb{T})$ we have $\left|\hat{f}(k)\right| \leq \frac{1}{|2\pi k|^{2j}} \left\|\triangle^j f\right\|_{L^\infty(\mathbb{T})}$.

(2) Assume now that $f \in C^\infty(\mathbb{T})$. Show that $F^{(\alpha)}(x) = \sum_{k\in\Lambda^*} (2\pi i k)^\alpha \hat{f}(k)e(kx)$ converges uniformly for all multi-indices α.

(3) Integrating term-by-term show that $F^{(\alpha)}$ is the αth derivative of $F^{(0)}$.
(4) Show that $F^{(0)} = f$ pointwise.

3. The Poisson Summation Formula

Definition. For $f \in L^1(V)$ and $k \in V^*$ set $\hat{f}(k) = \int_V f(x)e(-kx)dx$ and call this the *Fourier transform* of f.

Exercise 12 (The Fourier transform)**.** Let $f \in L^1(V)$

(1) Show that $\|\hat{f}\|_{L^\infty(V^*)} \leq \|f\|_{L^1(V)}$.
(2) Show that $\hat{f} \in C(V)$.
Hint: The bounded convergence theorem.
(3) On $V = \mathbb{R}$ let $f = \exp(-|x|)$. Show that $\hat{f}(k) = \frac{2}{1+4\pi^2k^2}$.
(4) Let $\Re(\alpha) > 0$ and let $f(x) = \exp\left\{-\pi\alpha x^2\right\}$. Show that $\hat{f}(k) = \sqrt{\frac{1}{\alpha}} \exp\left\{-\frac{\pi}{\alpha}k^2\right\}$ where we take the branch of the square root with a cut at $(-\infty, 0]$.
Hint: Shift contours to reduce the problem to the known formula $\int_{\mathbb{R}} \exp\left(-\alpha x^2\right) dx = \sqrt{\frac{\pi}{\alpha}}$.
(5) Let $Q \in M_n(\mathbb{R})$ be a positive-definite symmetric matrix, and let $f(x) = \exp\left(-2\pi \langle x|Q|x\rangle\right)$. Show that $\hat{f}(k) = 2^{-n/2} (\det Q)^{-1/2} \exp\left\{-2\pi \left\langle k\left|Q^{-1}\right|k\right\rangle\right\}$.

We now prove our main theorem.

Exercise 13 (The Poisson Summation Formula)**.** Let $f \in C(\mathbb{R}^n)$ decay quickly enough.

(1) For $k \in \Lambda^*$ show that $\widehat{\Pi_\Lambda f}(k) = \frac{1}{\mathrm{covol}(\Lambda)} \hat{f}(k)$ where the first hat is the Fourier transform on $\mathbb{T}$ and the second is the one on V.

- Show that $\Pi_\Lambda f(x) = \frac{1}{\mathrm{covol}(\Lambda)} \sum_{\Lambda^*} \hat{f}(k)e(kx)$. Conclude that:

$$\sum_{v \in \Lambda} f(v) = \frac{1}{\mathrm{vol}(\Lambda)} \sum_{k \in \Lambda^*} f(k).$$

4. The Fourier transform and Fourier inversion on $\mathbb{R}^n$

Exercise 14 (Convolution)**.** For functions f, g on V set $(f * g)(x) = \int_V f(x+y)g(y)\,\mathrm{d}y$ if the integral converges absolutely.

(1) Show that $g * f = f * g$ whenver either is defined, and that the operation is bilinear, commutative and associative where defined.
(2) Show that $\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$, and conclude that convolution turns $L^1(V)$ into an algebra.
(3) Let $f, g \in L^1(V)$. Show that $\widehat{f * g} = \hat{f}\hat{g}$.

Exercise 15 (Smoothness vs decay)**.** (1) Suppose that f and all its partial derivatives up to order j belong to $L^1(V)$. Show that for j large enough (depending on N), $\left|\hat{f}(k)\right|$ decays polynomially at rate N.

(2) (Riemann–Lebesgue Lemma) Using the density of $C_c^\infty(V)$ in $L^1(V)$, show that for $f \in L^1(V)$, $\hat{f}$ *decays at infinity*: for every $\varepsilon > 0$ there is a compact set outside of which $\left|\hat{f}(k)\right| < \varepsilon$.
(3) Suppose that f decays polynomially. Show that for every j there is N such that if $|f(x)| \leq (1+|x|)^{-N}$ then $\hat{f} \in C^j(V^*)$.
(4) Suppose that $f \in C_c^\infty(V)$. In the integral defining $\hat{f}$, allow k to range over the complexified dual $\mathbb{C} \otimes_{\mathbb{R}} V^*$. Show that $\hat{f}$ extends to an entire function of k.

Exercise 16 (The Schwartz class and Fourier inversion)**.** Let $f \in \mathcal{S}(V)$

(1) Differentiating under the integral sign show that $\hat{f}(k)$ is smooth.
(2) Integrating by parts show that then $\hat{f}$ is of rapid decay.
(3) Combining the two calculations show that $\hat{f} \in \mathcal{S}(V)$.
(4) Applying the PSF to f with the lattice $r\Lambda$ and taking $r \to \infty$ show that

$$f(0) = \int_{V^*} \hat{f}(k)dk\,.$$

(5) Let $g(x) = f(x+y)$. Show that $\hat{g}(k) = \hat{f}(k)e(ky)$ and conclude that

$$f(x) = \int_{V^*} \hat{f}(k)e(kx)dk\,.$$

(6) Use the same methods to establish *Parseval's identity*: for $f \in \mathcal{S}(V)$,

$$\|f\|_{L^2(V)} = \left\|\hat{f}\right\|_{L^2(V^*)}\,.$$

(7) Conclude that the Fourier transform extends to a bijective isometry $\mathcal{F}\colon L^2(V) \to L^2(V^*)$, and that $\mathcal{F}^2$ is exactly reflection in the origin (the map that sends $f(x)$ to $f(-x)$).