Math 613: Problem set 1 (due 15/9/09)

Some number theory

1. For a commutative ring R write $R^\times$ for the group of invertible elements, $\mathrm{GL}_n(R)$ for the group $\{g \in M_n(R) \mid \det g \in R^\times\}$, and $\mathrm{SL}_n(R)$ for $\{g \in M_n(R) \mid \det g = 1\}$.
   (a) Show that $\mathrm{GL}_n(\mathbb{Z})$, $\mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z})$ are the automorphism groups of the additive groups of the rings $\mathbb{Z}^n$, $(\mathbb{Z}/N\mathbb{Z})^n$ respectively.
   OPT Show that $\mathrm{GL}_n(R)$ is the automorphism group of the R-module R^n.
   (b) Let N_1, N_2 be relatively prime and let $N = N_1N_2$. Show that $\mathrm{GL}_n(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{GL}_n(\mathbb{Z}/N_1\mathbb{Z}) \times \mathrm{GL}_n(\mathbb{Z}/N_2\mathbb{Z})$.
   (c) Show that the maps $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ (reduction mod N) are surjective.
   Hint: Given $\bar{\gamma} \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ choose a pre-image $\gamma \in M_2(\mathbb{Z})$ such that the entries in the bottom row of γ are relatively prime.
   (d) Find the image of the map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.
   Hint: What is $\mathbb{Z}^\times$?
   OPT Do parts (c),(d) for SL_n, GL_n.
   OPT Do parts (c),(d) replacing $\mathbb{Z}$ with the ring of integers of a number field and N with an ideal in the ring of integers.

2. Let G be a group, $H \operatorname{char} G$ a *characteristic* subgroup. In other words, one such that for every automorphism $\sigma \in \mathrm{Aut}(G)$ we have $\sigma(H) = H$.
   (a) Show $H \lhd G$.
   (b) Show that there is a natural map $\mathrm{Aut}(G) \to \mathrm{Aut}(G/H)$.
   *(c) Classify the orbits of $\mathrm{Aut}(\mathbb{Z}^2)$ on $\mathbb{Z}^2$.
   (d) Find all chracteristic subgroups of $\mathbb{Z}^2$.
   OPT Do parts (c),(d) in $\mathbb{Z}^n$.

Lattices in $\mathbb{R}^n$

3. (Construction) Let $\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$ be linearly independent, let $\Lambda = \left\{\sum_{j=1}^k a_j v_j \mid \underline{a} \in \mathbb{Z}^k\right\} \subset \mathbb{R}^n$ be the subgroup they generate, and let $\mathbb{R}^n/\Lambda$ be the quotient group, endowed with the quotient topology coming from the map $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\Lambda$.
   (a) Show that the map $\mathbb{Z}^k \to \Lambda$ given by $\underline{a} \to \sum_j a_j v_j$ is an isomorphism.
   (b) Show that Λ is a discrete subset of $\mathbb{R}^n$.
   (c) Given $x, y \in \mathbb{R}^n$ such that $\pi(x) \neq \pi(y)$ find open sets $U_x, U_y \subset \mathbb{R}^n$ containing x, y respectively such that $\pi(U_x) \cap \pi(U_y) = \emptyset$. You have shown that $\mathbb{R}^n/\Lambda$ is Hausdorff.
   Hint: Let $r = \min\{\|v\| \mid v \in \Lambda, v \neq 0\}$.
   (d) Show that $\mathbb{R}^n/\Lambda$ isn't compact if $k < n$.
   (e) Let $k = n$, and let $\mathcal{F} = \left\{\sum_{j=1}^k a_j v_j \mid \forall j : |a_j| \leq \frac{1}{2}\right\}$. Show that $\mathcal{F}$ surjects onto $\mathbb{R}^n/\Lambda$ and conclude that $\mathbb{R}^n/\Lambda$ is compact.
   HINT Applying an automorphism of $\mathbb{R}^n$ before starting the problem will make your life much easier.

4. (Reduction theory) Let $\Lambda \subset \mathbb{R}^n$ be a discrete subgroup. Set $\Lambda_0 = \{0\}$, $V_0 = \{0\}$ and for $j \geq 1$ if $\Lambda \not\subset V_{j-1}$ choose $v_j \in \Lambda \setminus V_{j-1}$ minimizing the distance to V_{j-1}. Then set $\Lambda_j = \Lambda_{j-1} + \mathbb{Z}v_j$, $V_j = V_{j-1} + \mathbb{R}v_j$.
   (a) Assume by induction that $\Lambda_{j-1} = \Lambda \cap V_{j-1}$ and that it is a lattice in V_{j-1}. Show that set of distances $\left\{d(v, V_{j-1})\right\}_{v \in \Lambda}$ has a minimal non-zero member, so that v_j exists.
   Hint: Consider first the set of distances $d(v, V_{j-1})$ for vectors v whose orthogonal projection to V_{j-1} lies in $\mathcal{F}_{j-1} = \left\{\sum_{i=1}^{j-1} a_i v_i \mid |a_i| \leq \frac{1}{2}\right\}$.
   (b) Show that $\Lambda_j = \Lambda \cap V_j$.
   (c) Conclude that $\Lambda = \mathbb{Z}v_1 \oplus \cdots \mathbb{Z}v_j$ for some $0 \leq j \leq n$.

DEFINITION. Call $\Lambda < \mathbb{R}^n$ a *lattice* if it is discrete and if $\mathbb{R}^n/\Lambda$ is compact.

Convergence Lemma

Write $B(R)$ for the closed ball of radius R in $\mathbb{R}^n$, c_n for the volume of $B(1)$ so that $\operatorname{vol}(B(R)) = c_n R^n$. Fix a lattice $\Lambda < \mathbb{R}^n$.

5. Show that there exist $V, C > 0$ such that for any $R \geq 1$,
$$|\#(\Lambda \cap B(R)) - VR^n| \leq CR^{n-1}.$$
   Hint: Consider the set $\bigcup_{v \in \Lambda \cap B(R)} (v + \mathcal{F})$, and prove the claim first for $R \geq 2\operatorname{diam}(\mathcal{F})$.

6. For $s \in \mathbb{C}$ the *Epstein zetafunction* is given by
$$E(\Lambda; s) = \sum_{v \in \Lambda}{}' \|v\|^{-ns},$$
   where the prime indicates summation over non-zero elements of Λ.
   (a) Show that the series defining $E(\Lambda; \sigma)$ converges for any real $\sigma > 1$.
   Hint: You can use 5, or the identity $\int_{\mathbb{R}^n} f(x)dx = \sum_{v \in \Lambda} \int_{v+\mathcal{F}} f(x)dx$.
   (b) Show that the series defining $E(\Lambda; s)$ converges uniformly absolutely in any right half-plane of the form $\Re(s) \geq \sigma > 1$.
   (c) Conclude that the series defines a holomorphic function in the open half-plane $\Re(s) > 1$.
   (d) For $n = 1$ relate $E(\Lambda; s)$ to the Riemann zetafunction.

REMARK. In the next problem set we will analytically continue $E(\Lambda; s)$, showing that it extends to a meromorphic function on $\mathbb{C}$ bounded in vertical strips with poles at $0, 1$ and satisfying a functional equation relating the values at s and $1 - s$.

Later in the course we will also fix s and consider $E(\Lambda; s)$ as a function of Λ.

Extra: The "moduli space of complex annuli"

8. Given $0 < r < s$ let Let $A_{r,s} = \{z \in \mathbb{C} \mid r < |z| < s\}$. Write A_r for $A_{r,1}$. Show that $A_{r,s}$ and $A_{r',s'}$ are biholomorphic when $r'/s' = r/s$.

9. Let $f\colon A_r \to A_{r'}$ be a biholomorphism.
   (a) Show that as $z \to \partial A_r$, $f(z) \to \partial A_{r'}$.
   (b) Show that for $\varepsilon > 0$ and all small enough δ (depending on ε), $f(A_{r+\delta,1-\delta}) \supset A_{r'+\varepsilon,1-\varepsilon}$. Conclude that, up to inversion, we have $|f(z)| \xrightarrow[|z|\to 1]{} 1$ and $|f(z)| \xrightarrow[|z|\to r]{} r'$.
   (c) Let $g(z) = \log r \log |f(z)| - \log r' \log |z|$. Show that g is harmonic in A_r and vanishes at ∂A_r. Conclude that $g(z) = 0$.
   (d) Show that $f(z) = cz$ where $|c| = 1$, and hence that $r = r'$.