Math 312: Problem set 1 (due 13/5/11)

The factorial function and binomial coefficients

Recall that the factorial function is defined by $0! = 1$ and for $n \geq 0$ by $(n+1)! = (n+1) \cdot n!$. The *binomial coefficents* are defined for $0 \leq k \leq n$ by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n \geq 0$ we set $\binom{n}{k} = 0$ (for example, $\binom{4}{2} = 6$ while $\binom{2}{4} = 0$).

1. For $0 \leq k < n$ show that $\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1}$ by a direct calculation.
   OPT: show that this holds even if $0 \leq n \leq k$.

2. For $n \geq 0$ show that $\sum_{j=0}^{n} \binom{j}{0} = \binom{n+1}{1}$.
   Hint: once you unwind the definitions of both sides this is not hard.

Induction

Use mathematical induction to prove the following assertions:

3. Among every three consecutive positive integers there is one that is divisible by 3.

4. Fix $k \geq 0$ and show by induction on n that for $n \geq 1$, $\sum_{j=0}^{n-1} \binom{j}{k} = \binom{n}{k+1}$.
   Hint: Use the conclusion of problem 1, including the optional part

5. (Summation formulas)
   (a) Show that $j^2 = 2\binom{j}{2} + \binom{j}{1}$. This means that $\sum_{j=0}^{n} j^2 = 2\sum_{j=0}^{n} \binom{j}{2} + \sum_{j=0}^{n} \binom{j}{1}$ (why?). Use problem 4 to establish a formula for $\sum_{j=0}^{n} j^2$.
   Hint: You can check your formula (but not the proof) using §1.3.E7.
   (b) Express j^3 as a combination of $\binom{j}{3}$, $\binom{j}{2}$, $\binom{j}{1}$ and use problem 4 to prove a formula for $\sum_{j=0}^{n} j^3$.
   Hint: Check your formula against §1.3.E22.

Divisibility

An integer a is said to *divide* the integer b if there is a third integer c such that $ac = b$. For example, 2 divides 6 since $2 \cdot 3 = 6$, but 5 does not divide 6.

6. For each integer $n \in \{6, 12, 17\}$:
   (a) List the positive integers which divide n.
   (b) Find the sum of the divisors of n which are different from n (that is, for each n add all the numbers you got in part (a) except for n itself).
   (c) Is n *abundant* (the sum is bigger than n), *deficient* (the sum is less than n) or *perfect* (the sum is equal to n)?
7. Using the lists of divisors from the previous problem:
   (a) What is the largest number that divides both 6 and 12?
   (b) What is the largest number that divides both 12 and 17?

REMARK. Perfect numbers are rare and only finitely many are known. It is believed that there are infinitely many even perfect numbers, but this is not known. It is not known if there exist any odd perfect numbers.

Factorials and primes

8. For $2 \leq j \leq n$. Show that $n! + j$ is not prime. Conclude that there are arbitrarily large gaps between consecutive primes.

9. For which prime numbers p is $p+1$ also prime?

REMARK. It is believed (the "twin prime conjecture") that there are infinitely many primes p for which $p+2$ is also prime.