Math 100 §105, Fall Term 2010
Solutions to Midterm Exam

October 4$^{\text{th}}$,2010

Student number:

LAST name:

First name:

Instructions

- Do not turn this page over until instructed. You will have 45 minutes for the exam.
- You may not use books, notes or electronic devices of any kind.
- Solutions should be written clearly, in complete English sentences, showing all your work.
- If you are using a result from the textbook, the lectures or the problem sets, state it properly.

Signature:

1		/18
2		/8
3		/4
4		/10
Total		/40

1 Short-form answers

Show your work and clearly delineate your final answer. Not all problems are of equal difficulty.

[3] a. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to\infty} \frac{x^3 - \sin x}{2x^3 + 5x + 1}$$

$\lim_{x\to\infty} \frac{x^3-\sin x}{2x^3+5x+1} = \lim_{x\to\infty} \frac{1-x^{-3}\sin x}{2+5x^{-2}+x^{-3}}$. Now $-\frac{1}{x^3} \le \frac{\sin x}{x^3} \le \frac{1}{x^3}$ so $\lim_{x\to\infty} \frac{\sin x}{x^3} = 0$ by the squeeze theorem. The quotient rule and linearity give

$$\lim_{x\to\infty} \frac{1 - x^{-3}\sin x}{2 + 5x^{-2} + x^{-3}} = \frac{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{\sin x}{x^3}}{2 + 5\lim_{x\to\infty} x^{-2} + \lim_{x\to\infty} x^{-3}} = \frac{1-0}{2+5\cdot 0+0} = \frac{1}{2}\,.$$

[3] b. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to 0} \frac{\tan x}{x}$$

By the quotient rule, $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sin x}{x\cos x} = \frac{\lim_{x\to 0} \frac{\sin x}{x}}{\lim_{x\to 0} \cos x} = \frac{1}{1} = 1$ since $\lim_{x\to 0} \frac{\sin}{x} = 1$ from the lecture and $\lim_{x\to 0} \cos x = \cos(0) = 1$ by continuity.

[3] c. Evaluate the following limit (or show it does not exist):

$$\lim_{x\to 0} \frac{\sqrt{1+2x^2} - \sqrt{1+x^2}}{x^3}$$

We have $\frac{\sqrt{1+2x^2}-\sqrt{1+x^2}}{x^3} = \frac{\sqrt{1+2x^2}-\sqrt{1+x^2}}{x^3} \cdot \frac{\sqrt{1+2x^2}+\sqrt{1+x^2}}{\sqrt{1+2x^2}+\sqrt{1+x^2}} = \frac{(1+2x^2)-(1+x^2)}{x^3} \cdot \frac{1}{\sqrt{1+2x^2}+\sqrt{1+x^2}} = \frac{x^2}{x^3} \cdot \frac{1}{\sqrt{1+2x^2}+\sqrt{1+x^2}} = \frac{1}{x} \cdot \frac{1}{\sqrt{1+2x^2}+\sqrt{1+x^2}}$. Now $\frac{1}{\sqrt{1+2x^2}+\sqrt{1+x^2}}$ is a continuous function, so $\lim_{x\to 0} \frac{1}{\sqrt{1+2x^2}+\sqrt{1+x^2}} = \frac{1}{1+1} = \frac{1}{2} \neq 0$ while $\frac{1}{x}$ grows without bound. It follows that the given limit *does not exist*.

[3] d. Differentiate the following function:

$$\left(1 + x^2 \sin x\right)^{1/3}$$

By the chain rule, $\frac{d}{dx}\left(1 + x^2 \sin x\right)^{1/3} = \frac{1}{3}\left(1 + x^2 \sin x\right)^{-2/3} \cdot \frac{d}{dx}\left(1 + x^2 \sin x\right)$. We evaluate the latter derivative by the product rule and find

$$\frac{d}{dx}\left(1 + x^2 \sin x\right)^{1/3} = \frac{1}{3}\left(1 + x^2 \sin x\right)^{-2/3}\left(2x \sin x + x^2 \cos x\right).$$

[3] e. Differentiate the following function:

$$\frac{e^x + e^{-x}}{2\cos x}$$

By the quotient rule this is

$$\frac{1}{2}\frac{\left(e^x - e^{-x}\right)\cos x + \left(e^x + e^{-x}\right)\sin x}{\cos^2 x}.$$

[3] f. Write an equation of the form $y = ax + b$ for the line tangent to the following function at the point $(1, 1)$.

$$y = x^4 - \frac{1}{\pi}\sin(\pi x)$$

Since $\frac{dy}{dx} = 4x^3 - \cos(\pi x)$ the slope of the tangent at $x = 1$ is $4 - \cos(\pi) = 4 - (-1) = 5$. The equation for the line is then $y - 1 = 5(x - 1)$, that is

$$y = 5x - 4.$$

2 Long-form answers

A ball falling from rest in air is at height $h(t) = H_0 - gt_0(t + t_0e^{-t/t_0} - t_0)$ at time t. Here H_0 is the initial height, g is the gravitational constant and t_0 depends on the body.

[3] a. Find the velocity $v(t)$ of the ball.
We have $h(t) = H_0 + gt_0^2 - gt_0t - gt_0^2e^{-t/t_0}$. The first two terms are constant, and the the third is linear. For the last term we use the chain rule to find $\left(e^{-t/t_0}\right)' = \left(-\frac{1}{t_0}\right)e^{-t/t_0}$ so the derivative is:

$$v(t) = \frac{dh}{dt} = -gt_0 - gt_0^2e^{-t/t_0}\left(-\frac{1}{t_0}\right) = gt_0\left(e^{-\frac{t}{t_0}} - 1\right).$$

Common difficulties:

1. Not realizing that t_0 is a constant.
2. When using the chain rule, getting $-\frac{t}{t_0}e^{-t/t_0}$ (note the extra factor of t).
3. Many sign errors.

[2] b. Find its acceleration $a(t)$.

We differentiate again to find

$$a(t) = \frac{dv}{dt} = gt_0\left(-\frac{1}{t_0}e^{-\frac{t}{t_0}} - 0\right) = -ge^{-t/t_0}.$$

Common difficulties:

1. Sign errors.
2. If made error (2) above, not using the product rule to differentiate $t \cdot e^{t/t_0}$.

[1] c. Find $v(0)$, $a(0)$.
$v(0) = gt_0\left(e^{-0} - 1\right) = gt_0(1-1) = 0$ and $a(0) = -ge^{-0} = -g$.

[2] d. Find $\lim_{t\to\infty} v(t)$.
As $t \to \infty$, t/t_0 tends to ∞. Hence so does e^{t/t_0}. It follows that $\lim_{t\to\infty} e^{-t/t_0} = \lim_{t\to\infty} \frac{1}{e^{t/t_0}} = 0$. By linearity of the limit we find

$$\lim_{t\to\infty} v(t) = gt_0\,(0-1) = -gt_0\,.$$

Common difficulties:

1. Writing things like e^{∞} and $e^{-\infty}$ which don't make sense.
2. Claiming that $\lim_{t\to\infty} e^{-t/t_0} = \infty$.

3 Long-form answers

[4] Let $f(x)$ be a function defined for $0 \le x \le 10$. You are given that $f(5) = 1$ and that $f'(5)$ exists and equals 8. Using only the definition of the derivative, evaluate $h'(5)$ where $h(x) = (f(x))^2$.

We need to evaluate

$$\begin{aligned}
\lim_{t\to 0} \frac{h(5+t)-h(5)}{t} &= \lim_{t\to 0} \frac{(f(5+t))^2 - (f(5))^2}{t} \\
&= \lim_{t\to 0} \frac{(f(5+t) - f(5)\,(f(5+t)+f(5))}{t} \\
&= \lim_{t\to 0} \left[\frac{(f(5+t)-f(5))}{t} \cdot (f(5+t)+f(5))\right] \\
\text{(product rule)} &= \lim_{t\to 0} \left[\frac{(f(5+t)-f(5))}{t}\right] \cdot \lim_{t\to 0} [f(5+t)+f(5)] \\
\text{(definition of } f'(5) + \text{linearity)} &= f'(5) \cdot \left[\left(\lim_{x\to 5} f(x)\right) + f(5)\right] \\
\text{(the function is continuous where differentiable)} &= f'(5)\,[f(5)+f(5)] \\
&= 2f(5)f'(5) \\
&= 2 \cdot 8 \cdot 1 = 16\,.
\end{aligned}$$

Common difficulties:

1. Solving the problem by the chain rule. The problem explicitly says not to do that.
2. This was by far the hardest problem; only 4 correct solutions among 200 students. Nearly everyone just got 1 point for knowing the definition of the derivative.
3. Many people wrote things like $(f(x+t))^2 = f(x)^2 + 2f(x)t + t^2$ or $(f(x+t))^2 = f\left((x+t)^2\right) = f(x^2 + 2xt + t^2)$. This is a sign of doing symbolic manipulation without meaning.
4. Some students hit on the (good) idea of first approximating f by its tangent line at $x = 5$, and then calculating the limit. This gives the right answer (by the chain rule), but is not quite a correct solution.

4 Long-form answers

The function $f(x)$ is defined for non-zero x by

$$f(x) = \begin{cases} ax^2 + bx + c & x < 0 \\ 2 + x^3 \cos(x^{-1}) & x > 0 \end{cases}.$$

[5] a. Determine all values (if any exist) of the constants a, b, c so that $f(x)$ can be made continuous for all x by choosing $f(0)$ appropriately. (Don't forget to justify your answer!)

Note first that f is continuous for both $x < 0$ and $x > 0$ so we only need to check at $x = 0$. We have $\lim_{x\to 0^-} f(x) = \lim_{x\to 0} \left(ax^2 + bx + c\right) = c$ since polynomials are continuous. On the other side $\cos(x^{-1})$ is bounded between -1 and 1 while x^3 tends to zero so the squeeze theorem shows $\lim_{x\to 0} x^3 \cos(x^{-1}) = 0$ and hence $\lim_{x\to 0^+} f(x) = 2$. To make f continuos at 0 we need both one-sided limits to equal $f(0)$ so we need both to be equal, in which case we set $f(0)$ equal to that value. To conclude, f can be made continuous exactly when $c = 2$ (and a, b can be arbitrary).

Common difficulties:

1. The most common way to calculate $\lim_{x\to 0^+} f(x)$ was to write: $f(0) = 2 + 0^3 \cos(0^{-1}) = 2$. Unfortunately 0^{-1} and $\cos(0^{-1})$ are not numbers.
2. A large minority wrote something like $ax^2 + bx + c = 2 + x^3 \cos(x^{-1})$ and tried to continue from there. But each of the two expression represents f for *different values of x*.
3. Others said that this equality should hold "at $x = 0$". But the problem does not define f at zero. What should be true is that the *limits* at zero are equal.
4. Some students tried to relate the derivatives of the two expressions.

[5] b. Choosing $f(0)$ as above, determine all values (if any exist) of the constants a, b, c so that $f'(x)$ is continuous for all x. (Don't forget to justify your answer!)

For $x < 0$ we have $f'(x) = 2ax + b$ which is continuous. We also note $\lim_{x\to 0^-} f'(x) = b$. For $x > 0$ we have $f'(x) = (x^3)' \cos(x^{-1}) - x^3 \left(\cos(x^{-1})\right)' = 3x^2 \cos(x^{-1}) + x^3 \left(-\sin(x^{-1})\right)\left(-\frac{1}{x^2}\right) = 3x^2 \cos(x^{-1}) + x \sin(x^{-1})$ so $f'(x)$ is continuous for $x > 0$ also. Since both $\cos(x^{-1}), \sin(x^{-1})$ are bounded while $x^2, x^3 \to 0$ as $x \to 0$ we also have $\lim_{x\to 0^+} f'(x) = 0$. To make f' continuous at $x = 0$ we then need $b = 0$, and we still need $c = 2$ (if $f'(0)$ is to exist we need f continuous!), but a can be arbitrary.

Common difficulties:

1. Differentiating $x^3 \cos(x^{-1})$ correctly was difficult. Answers included: $3x^2(-\sin(x^{-1}))$ (differentating both factors and $3x^2 \cos(x^{-1}) - x^3 \sin(x^{-1})$.
2. Again, $\sin(0^{-1})$ and $\cos(0^{-1})$ don't make sense.
3. Forgetting to put parathenses around factors with a minus sign makes a difference. $x^3 - \frac{1}{x^2}$ is not the usual notation for the proudct of x^3 and $-\frac{1}{x^3}$, but for the difference of x^3 and $\frac{1}{x^2}$.